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Appendix. Formal Theories of Parthood

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This Appendix gives a brief overview of the main formal theories of parthood, or mereologies, to be found in the literature.[1] The focus is on classical theories, so the survey is not meant to be exhaustive. Moreover, it does not cover the many philosophical issues relating to the endorsement of the theories themselves, concerning which the reader is referred to the Selected Bibliography at the end of the volume. In particular, we shall be working under the following simplifying assumptions:[2]

— *Absoluteness*: Parthood is a two-place relation; it does not hold relative to time, space, spacetime regions, sortals, worlds, or anything else.[3]

— *Monism*: There is a single relation of parthood that applies to every entity independently of its ontological category.[4]

— *Precision*: Parthood is not a source of vagueness: there is always a fact of the matter as to whether the parthood relation obtains between any given pair of things.[5]

For definiteness, all theories will be formulated in a standard first-order language with identity, supplied with a distinguished binary predicate constant, '*P*', to be interpreted as the parthood relation. The underlying logic will be the classical predicate calculus.

1 Core Principles

As a minimal requirement on '*P*', it is customary to assume that it stands for a partial order—a reflexive, transitive, and antisymmetric relation:[6]

(P.1) Pxx *Reflexivity*
(P.2) $(Pxy \wedge Pyz) \rightarrow Pxz$ *Transitivity*
(P.3) $(Pxy \wedge Pyx) \rightarrow x = y$ *Antisymmetry*

Together, these three axioms are meant to fix the intended meaning of the parthood predicate. They form the "core" of any standard mereological theory, and the theory

[1] The exposition follows Varzi (2014). For a thorough survey, see Simons (1987).
[2] The labels and formulations of these assumptions are from Sider (2007).
[3] For the view that parthood should be a three-place relation relativized to time, see e.g. Thomson (1983). For the view that it should be a four-place relation, see Gilmore (2009).
[4] For misgivings about *Absoluteness* and related worries, see e.g. Mellor (2006) and McDaniel (2009).
[5] For mereologies that allow for indeterminate or "fuzzy" parthood relations, see e.g. Smith (2005) and Polkowski (2011).
[6] Unless otherwise specified, all formulas are to be understood as universally closed.

that comprises just them is called *Ground Mereology*, or M for short.[7] A number of additional mereological predicates may then be introduced by definition:

(1)	$EQxy =_{df} Pxy \wedge Pyx$		*equality*
(2)	$PPxy =_{df} Pxy \wedge \neg x = y$		*proper parthood*[8]
(3)	$PExy =_{df} Pyx \wedge \neg x = y$		*proper extension*
(4)	$Oxy =_{df} \exists z\,(Pzx \wedge Pzy)$		*overlap*
(5)	$Uxy =_{df} \exists z\,(Pxz \wedge Pyz)$		*underlap*

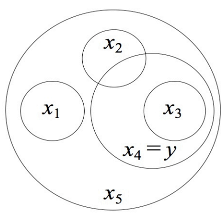
Given (P.1)–(P.3), it follows immediately that *EQ* is an equivalence relation. Moreover, *PP* and *PE* are irreflexive, asymmetric, and transitive whereas *O* and *U* are reflexive and symmetric, but not transtive. Since the following is a also theorem of M,

(6) $Pxy \leftrightarrow (PPxy \vee x = y)$

'*PP*' could have been used as a primitive instead of '*P*'. Similarly for '*PE*'. Sometimes '*P*' is also defined in terms of '*O*' via the biconditional

(7) $Pxy \leftrightarrow \forall z(Ozx \rightarrow Ozy)$.

However, (7) is not provable in M and calls for stronger axioms (specifically, the axioms of theory EM defined below). Since those stronger axioms reflect substantive philosophical theses, '*P*' and '*PP*' (or '*PE*') are the best options to start with. Here we stick to '*P*'.



	Ux_ny	Ox_ny	PPx_ny	EQx_ny	PEx_ny
$n = 1$	+	–	–	–	–
$n = 2$	+	+	–	–	–
$n = 3$	+	+	+	–	–
$n = 4$	+	+	–	+	–
$n = 5$	+	+	–	–	+

Figure 1. Basic patterns of mereological relations. (Shaded cells indicate parthood).

2 Decomposition Principles

M is standardly viewed as embodying the common core of any mereological theory. Yet not every partial order qualifies as parthood, and establishing what further re-

[7] For a survey of the motivations that may lead to the development of non-standard mereologies in which *P* is weaker than a partial order, see Varzi (2014: §2.1).

[8] In the literature, proper parthood is sometimes defined as asymmetric parthood: $PPxy =_{df} Pxy \wedge \neg Pyx$. Given *Antisymmetry*, this definition is equivalent to (2). Without *Antisymmetry*, however, the two definitions would come apart. (Similarly for '*PE*'.) See Cotnoir (2010).

quirements should be added to (P.1)–(P.3) is precisely the question a good mereological theory is meant to answer.

One way to extend M is by means of *decomposition principles*, i.e., principles concerning the part structure of a given whole. Here, one fundamental intuition is that no whole can have a single proper part. There are several ways in which this intuition can be captured, beginning with the following:

(P.4_a) $PPxy \rightarrow \exists z(PPzy \wedge \neg z = x)$ *(Weak) Company*
(P.4_b) $PPxy \rightarrow \exists z(PPzy \wedge \neg Pzx)$ *Strong Company*
(P.4) $PPxy \rightarrow \exists z(PPzy \wedge \neg Ozx)$ *(Weak) Supplementation*[9]

(P.4_a) is the literal rendering of the idea in question, but it is too weak: it rules out certain implausible finitary models (Fig. 2, left) but not, for example, models with infinitely descending chains in which the additional parts do not leave any mereological "remainder" (Fig. 2, center). (P.4_b) is stronger, but it still admits of models in which a whole can be decomposed into several proper parts all of which overlap one another (Fig. 2, right). In such cases it is unclear what would be left of the whole upon the removal of any of its proper parts (along with all proper parts thereof). It is only (P.4) that appears to capture the full spirit of the above-mentioned intuition: every proper part must be "supplemented" by another part—a proper part that is completely disjoint (i.e., does not overlap) the first. (P.4) entails both (P.4_a) and (P.4_b) and rules out each of the models in Fig. 2. The extension of M obtained by adding this principle to (P.1)–(P.3) is called *Minimal Mereology*, or MM.[10]

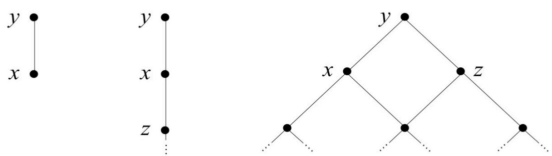


Figure 2. Three unsupplemented models. (Connecting lines going upwards indicate proper parthood.)

There is another, stronger way of expressing the supplementation intuition. It corresponds to the following axiom, which differs from (P.4) in the antecedent:

(P.5) $\neg Pyx \rightarrow \exists z(Pzy \wedge \neg Ozx)$ *Strong Supplementation*

In M this principle entails (P.4). The converse, however, does not hold, as shown by

[9] In the literature, this principle is sometimes formulated using '*P*' in place of '*PP*' in the consequent. In M the two formulations are equivalent.
[10] Strictly speaking, in MM (P.3) is redundant, as it follows from (P.4) along with (P.1) and (P.2). For ease of reference, however, we shall continue to treat (P.3) as an axiom.

the model in Fig. 3. The stronger mereological theory obtained by adding (P.5) to the three core principles of M is called *Extensional Mereology*, EM.

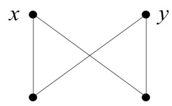


Figure 3. A weakly supplemented model violating strong supplementation.

The extensional character of EM may not be manifest in (P.5) itself, but it becomes clearer in view of the following theorem:

(8) $\exists zPPzx \rightarrow (\forall z(PPzx \rightarrow PPzy) \rightarrow Pxy)$

from which it follows that sameness of mereological composition is both necessary and sufficient for identity:

(9) $\exists zPPzx \rightarrow (x{=}y \leftrightarrow \forall z(PPzx \leftrightarrow PPzy))$.

Thus, EM is "extensional" precisely insofar as it rules out *any* model of the sort depicted in Fig. 3, where distinct objects decompose into the same proper parts.

There is yet a further way of capturing the supplementation intuition. It corresponds to the following axioms, which differs from (P.5) in the consequent:

(P.6) $\neg Pyx \rightarrow \exists z\forall w(Pwz \leftrightarrow (Pwy \wedge \neg Owx))$. *Complementation*[11]

Informally, (P.6) states that whenever an object fails to include another among its parts, there is something that amounts exactly to the *difference* or relative *complement* between the first object and the second. Once again, it is easily checked that in M this principle entails (P.5)—thus, *a fortiori*, (P.4)—whereas the converse does not hold (Fig. 4).

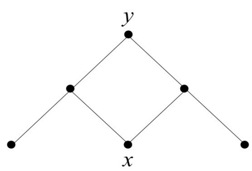


Figure 4. A strongly supplemented model violating complementation.

It should be noted, however, that (P.6) goes beyond the original supplementation intuition. For while it guarantees that a whole cannot have a single proper part, it

[11] In the literature, (P.6) is also known as the *Remainder Principle*.

also pronounces on the specific mereological makeup of the supplementary part. In particular, it requires the relative complement to exist regardless of its internal structure. If, for example, *y* is a wine glass and *x* the stem of the glass, (P.6) entails the existence of something composed exactly of the base and the bowl—a spatially disconnected entity. Whether there exist entities of this sort, and more generally whether the remainder between a whole and any one of its proper parts adds up to a *bona fide* entity of its own, is really a question about mereological composition, over and above the conditions on decomposition set by (P.4) and (P.5).

Before turning to issues regarding composition, a different sort of decomposition principles is worth mentioning. Let a mereological atom be any entity with no proper parts:

(10) $A\mathrm{x} =_{\mathrm{df}} \neg\exists yPPyx.$ *atom*

Obviously, all the theories considered so far are compatible with the existence of such things. But one may want to demand more than mere compatibility, just as one may want to preclude it. Thus, one may want to require that everything is ultimately composed of atoms, or else that everything is made up of "atomless gunk"[12] that divides forever into smaller and smaller parts. These two options are usually formulated as follows:

(P.7) $\exists y(Ay \wedge Pyx)$ *Atomicity*
(P.8) $\exists yPPyx$ *Atomlessness*

These postulates are mutually inconsistent, but taken in isolation they can consistently be added to any mereological theory mentioned so far to yield either an *atomistic* variant or an *atomless* variant, respectively.

Atomistic mereologies admit significant semplifications in the axiomatics. For example, Atomistic EM can be simplified by merging *Strong Supplementation* (P.5) and *Atomicity* (P.7) into a single axiom:

(P.5′) $\neg Pxy \rightarrow \exists z(Az \wedge Pzx \wedge \neg Pzy)$ *Atomistic Supplementation*

and the the extensionality thesis (9) can be put more perspicuously as follows:

(9′) $x{=}y \leftrightarrow \forall z(Az \rightarrow (Pzx \leftrightarrow Pzy))$

This is especially significant if one considers that (P.7) does not quite say that everything is *made up* of atoms; it merely says that everything *has* atomic parts, which is consistent with the possibility of infinitely descending chains of decomposition that

[12] The phrase is from Lewis (1991: 20).

never bottom out (Fig. 5). Whether stronger versions of (P.7) can be formulated that rule out such dubious patterns is, at the moment, a question that has not been fully explored.[13]

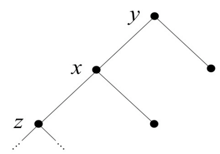


Figure 5. An infinitely descending atomistic model.

Concerning atomless mereologies, one may similarly remark that (P.8) is by itself rather weak. For one thing, the unsupplemented model in Fig. 2, middle, qualifies as atomless. To the extent that such models run afoul of the intended notion of a gunky world, this means that (P.8) calls for teories at least as strong as MM, in which case the relevant axiomatization may again be simplified by merging (P.4) and (P.8) into a single axiom:

(P.4′) $Pxy \rightarrow \exists z(PPzy \wedge (Ozx \rightarrow x = y))$ *Atomless Supplementation*

Moreover, infinite divisibility is loose talk. Given (P.8), gunk may have as few as denumerably many parts; but can it have more? Is there an upper bound on the cardinality on the number of pieces of gunk? Should it be allowed that for *every* cardinal number there may be more than that many pieces of gunk? (P.8) is silent on these questions, yet these are certainly aspects of atomless mereology that deserve further scrutiny.

3 Composition Principles

The other main way of extending M is via *composition principles*, i.e. principles governing the behavior of *P* in the bottom-up direction: from the parts to the wholes that they compose. We have already seen that the *Complementation* axiom (P.6) is, in a way, a principle of this sort. Another such principle would be the dual of *Atomlessness*, to the effect that everything might be "worldless junk"[14] that composes forever into greater and greater wholes:

(P.9) $\exists y PPxy$ *Ascent*

Both (P.6) and (P.9) are consistent with any of the theories considered so far. They are, however, fairly strong principles, which reflect specific views on the overall

[13] See Cotnoir (2013) for some work in this direction.
[14] The phrase is from Schaffer (2010: 64).

mereological structure of the universe. More generally, it is customary to consider ways of extending M by means of composition principles that specify the *conditions* under which one or more things qualify as parts of a larger whole.[15]

The most basic principles of this sort have the following form, to the effect that for any pair of suitably related entities, i.e., any two entities satisfying a given provision ξ, there is something of which both are part—an underlapper:

(P.10) $\xi xy \rightarrow Uxz$ *ξ-Bound*

Such principles are quite weak. For example, regardless of how exactly ξ is construed, (P.10) is trivially satisfied in any model that includes a universal entity of which everything is part.

A stronger sort of requirement is that any pair of suitably related entities have a *minimal* underlapper, something composed of their parts and nothing else. There are at least three ways of formulating such a requirement, corresponding to three different ways of characterizing the relevant notion of a minimal underlapper, also known as a mereological *sum* of the two entities in question:[16]

(11_a) $S_a zxy =_{df} Pxz \wedge Pyz \wedge \forall w((Pxw \wedge Pyw) \rightarrow Pzw)$ *a-sum*[17]
(11_b) $S_b zxy =_{df} Pxz \wedge Pyz \wedge \forall w(Pwz \rightarrow (Oxw \vee Oyw))$ *b-sum*
(11_c) $S_c zxy =_{df} \forall w(Owz \leftrightarrow (Oxw \vee Oyw))$ *c-sum*

In M these three notions are pairwise distinct (Fig. 6), though they may coincide in the presence of further axioms. For instance, given *Strong Supplementation*, (11_b) and (11_c) are equivalent (though stronger that (11_a)), whereas in the presence of *Complementation* all three notions coincide so long as there is a universal entity: in that case, each sum of any two things is just the complement of the difference between the complement of one minus the other. (Such is the strength of (P.6)—a genuine cross between decomposition and composition principles.)

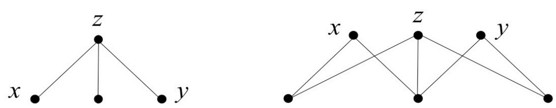


Figure 6. An a-sum that is not a b- or c-sum, and a c-sum that is not an a- or b-sum.[18]

15 This is a version of the so-called "Special Composition Question". See Van Inwagen (1990: Ch. 2).
16 The first notion may be found in Eberle (1967) and Bostock (1979), the second in Tarski (1935) and Lewis (1991), the third in Simons (1987) and Casati and Varzi (1999).
17 Given *Reflexivity* and *Transitivity*, the definiens in (11_a) is equivalent to $\forall w(Pzw \leftrightarrow (Pxw \wedge Pyw))$.
18 The non-extensional model of Fig. 3 also depicts a case in which *x* and *y* have a c-sum, in fact two c-sums (themselves), though no a- or b-sum. This runs contrary the intended meaning of 'sum', suggesting that (11_c) is best suited to theories at least as strong as EM. See Hovda (2009) for discussion.

For each i$\in$ {a, b, c}, we can then extend M by adding a corresponding axiom as follows, where again ξ specifies a suitable binary condition:

(P.11$_i$) $\xi xy \rightarrow \exists z S_i zxy$ *ξ-Sum$_i$*

The non-equivalence of these axioms is immediately verified by taking ξ to be satisfied by all pairs of objects and considering the models in Fig. 6. But the axioms may also differ when ξ is more restrictive. For instance, with ξ expressing overlap, the model in Fig. 6, right, still satisfies (P.11$_c$), but not (P.11$_a$) or (P.11$_b$), whereas the model in Fig. 7 satisfies (P.11$_a$), but not (P.11$_b$) or (P.11$_c$). In EM, however, (P.11$_b$) or (P.11$_c$) are equivalent, since the corresponding notions of sum coincide.

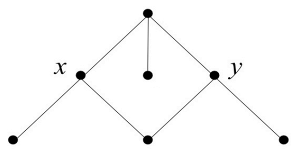


Figure 7. A model of ξ-sum$_a$ violating ξ-sum$_b$ and ξ-sum$_c$.

The intuitive force of each (P.11$_i$) is in fact best appreciated in the context of EM, for in that case the relevant sums must be unique. If we introduce a corresponding binary operator (using 'ι' for the definite descriptor),

(12$_i$) $x +_i y =_{df} \iota z F_i zxy$ *i-sum*

then is then easy to see that EM warrants all the "Boolean" properties one might expect. For instance, as long as the arguments satisfy the relevant condition ξ,[19] each operator is idempotent, commutative, and associative:

(13) $x = x +_i x$
(14) $x +_i y = y +_i x$
(15) $x +_i (y +_i z) = (x +_i y) +_i z$

and well-behaved with respect to parthood:

(16) $Px(x +_i y)$
(17) $Pxy \rightarrow Px(y +_i z)$
(18) $P(x +_i y)z \rightarrow Pxz$
(19) $Pxy \leftrightarrow x +_i y = y$

[19] If the condition is *not* satisfied, the sum may not exist, in which case the standard treatment of descriptive terms implies that the corresponding instances of the theorems that follow are false. In classical logic, (13)–(19) should therefore be taken to hold conditionally on the assumption that the relevant variables range over ξ-related entities.

Each (P.11$_i$) is still fairly weak, for it governs only finitary composition. We get even stronger composition principles by requiring a minimal underlapper to exist for *any* set of objects satisfying a given condition, including infinite sets (whose sums—or *fusions*—cannot be generated by means of the binary operators defined above). There is, of course, a technical obstacle to formulating such principles in their full generality without resorting to explicit quantification over sets, since a standard first-order language does not have the resources to specify all sets, but only a denumerable number (in any given domain).[20] However, one can achieve a sufficient degree of generality by relying on axiom *schemas* where the relevant sets are identified through open formulas. Thus, let 'φ' be any formula in the language, and let 'ψ' expresses the condition in question. Infinitary variants of the three notions of sum in (11$_a$)–(11$_c$) can be defined as follows, respectively:[21]

(20$_a$) $F_a z\varphi w =_{df} \forall w(\varphi w \rightarrow Pwz) \wedge \forall v(\forall w(\varphi w \rightarrow Pwv) \rightarrow Pzv)$ *a-fusion*
(20$_b$) $F_b z\varphi w =_{df} \forall w(\varphi w \rightarrow Pwz) \wedge \forall v(Pvz \rightarrow \exists w(\varphi w \wedge Owv))$ *b-fusion*
(20$_c$) $F_c z\varphi w =_{df} \forall v(Ovz \leftrightarrow \exists w(\varphi w \wedge Owv))$ *c-fusion*

('$F_i z\varphi w$' may be read as 'z is an i-fusion of the φ-ers'.) For each such notion, we may then introduce a corresponding principle of infinitary fusion through the following axiom schema, which asserts the existence of an i-fusion ($i \in \{a, b, c\}$) for every non-empty set of objects satisfying ψ:

(P.12$_i$) $(\exists w\varphi w \wedge \forall w(\varphi w \rightarrow \psi w)) \rightarrow \exists z F_i z\varphi w$ *ψ-Fusion$_i$*

It can be checked that each (P.12$_i$) includes the corresponding finitary principle (P.11$_i$) as a special case, taking 'φw' to be the formula '$w=x \vee w=y$' and 'ψw' the condition '$(w=x \rightarrow \xi wy) \wedge (w=y \rightarrow \xi xw)$'. Thus, again, these principles are pairwise distinct in M, though it turns out that in the presence of *Strong Supplementation* (P.12$_b$) and (P.12$_c$) are equivalent.

Finally, the strongest versions of all these composition principles are obtained by asserting them as axiom schemas holding for *every* condition ψ, i.e., effectively, by foregoing any reference to ψ altogether. Formally this amounts in each case to dropping the second conjunct of the antecedent of (P.12$_i$), i.e., to asserting the schema expressed by the relevant consequent for any non-empty set of objects specifiable in the language:

(P.13$_i$) $\exists w\varphi w \rightarrow \exists z F_i z\varphi w$ *Unrestricted Composition$_i$*

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[20] To overcome this limitation, some early theories such as those of Tarski (1929) and Leonard and Goodman (1940) resort to explicit quantification over sets. Others, such as Lewis (1991), resort to the machinery of plural quantification.

[21] (20$_a$)–(20$_c$) are to be read on the assumption that the variables 'z' and 'v' do not occur free in φ. Similar restrictions will apply below.

Once again, the relative strength of these principles varies for each $i \in \{a, b, c\}$. In particular, it is noteworthy that adding (P.13_b) to MM would suffice to warrant the equivalence of *Weak* and *Strong Supplementation*, (P.4) and (P.5), whereas adding (P.13_c) would not (Fig. 4 would still count as a counteremodel). Given (P.5), however, the two composition principles are equivalent, which means that the theory obtained by adding every instance of (P.13_b) to MM[22] is the same theory obtained by adding every instance of (P.13_c) to EM. This theory is known in the literature as *General Extensional Mereology*, or GEM. The same theory can be obtained by extending MM with (P.13_a), provided the following axiom is also added:[23]

(P.14) $(F_a z\varphi w \wedge Pxz) \to \exists w(\varphi w \wedge Owx)$ *Filtration*

4 Classical Mereology

GEM is a powerful theory, and it was meant to be so by its nominalistic forerunners, who were thinking of mereology as a fundamental alternative to set theory.[24] Indeed, GEM has such a distinguished pedigree that it has earned the title of *Classical Mereology*. It is also a decidable theory, whereas for example M, MM, EM, and many extensions thereof are not.[25] To see just how powerful GEM is, consider the following operator, where 'F' is any of the 'F_i's defined above (which GEM forces to coincide):

(21) $\sigma x\varphi x =_{df} \iota zFz\varphi x$ *general fusion*

In terms of this operator—the fusion of all φ-ers—GEM can be further simplified, for example by merging (P.5) and (P.13_c) into a single axiom schema:

(P.13) $\exists x\varphi x \to \exists z(z = \sigma x\varphi x)$ *Unique Unrestricted Fusion*

and we can introduce the following definitions:

(22) $x + y =_{df} \sigma z(Pzx \vee Pzy)$ *sum*[26]
(23) $x \times y =_{df} \sigma z(Pzx \wedge Pzy)$ *product*
(24) $x - y =_{df} \sigma z(Pzx \wedge \neg Ozy)$ *difference*
(25) $\sim x =_{df} \sigma z \neg Ozx$ *complement*
(26) $\mathrm{U} =_{df} \sigma zPzz$ *universe*

The full strength of GEM can then be appreciated by considering that its models are closed under each of these notions, subject to the satisfiability of the relevant condi-

[22] Indeed, (P.2) and (P.4) would suffice.
[23] From Hovda (2009).
[24] See the classical works of Leśniewski (1927–1931) and Leonard and Goodman (1940).
[25] For a comprehensive picture of decidability in mereology, see Tsai (2013).
[26] In GEM, this definition is equivalent to (12_i), for each $i \in \{a, b, c\}$.

tions. More exactly: the condition '$\neg OzU$' is unsatisfiable, so U cannot have a complement. Likewise products are defined only for overlappers and differences only for pairs that leave a remainder. In all other cases, however, (22)–(26) yield perfectly well-behaved operators. Since such operators are the natural mereological analogues of the familiar set-theoretic operators, with 'σ' in place of set abstraction, it follows that the parthood relation axiomatized by GEM has essentially the same properties as the inclusion relation in standard set theory, modulo the absence of a null entity corresponding to the empty set. Indeed, *P* is virtually isomorphic to the inclusion relation restricted to the set of all non-empty subsets of a given set, which is to say a complete Boolean algebra with the zero element removed. We say 'virtually' because, strictly speaking, this is only true of stronger version of GEM in which infinitary sums are defined using explicit quantification over sets.[27] For set-free formulations that, like those considered here, strictly adhere to a standard first-order language with a denumerable supply of open formulas, the isomorphism does not quite hold. However, this is only a minor limitation, and we can still characterize the exact algebraic strength of GEM in as follows: any model of this theory is isomorphic to a Boolean *subalgebra* of a complete Boolean algebra with the zero element removed (a subalgebra that is not necessarily complete if Zermelo-Frankel set theory with the axiom of Choice is consistent).[28]

In this connection, two further points are worth stressing. First, the existence of a "null entity" which is part of everything—the analogue of the empty set—is not in principle incompatible with GEM. However, it is easy to see that the only models of GEM with such an entity are trivial one-element models, owing to *Weak Supplementation*. It is for this reasons that the principles of *Unrestricted Composition* in (P.13_i) are stated as conditionals warranting the existence of a fusion for any given *non-empty* set of φ-ers. Dropping such a proviso would have disastrous effects, for then the existence of a null entity—*the* null entity—would be guaranteed by taking 'φw' to be the condition '$\forall xPwx$'. The only way around the disaster would be to revisit the non-basic vocabulary by carefully distinguishing trivial cases of parthood and overlap (involving the ubiquitous null entity) and non-trivial, genuine ones, as in

(27) $GPxy =_{df} Pxy \wedge \exists z \neg Pxz$ *genuine parthood*
(28) $GOxy =_{df} \exists z(GPzx \wedge GPzy)$ *genuine overlap*

and by reformulating all non-core axioms accordingly.[29] In this way, one can actually arrive at a variant of GEM that inherits all the strength of a complete Boolean algebra. Nonetheless, the philosophical import of such a theory would remain dubious.

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[27] As such, the result goes back to Tarski (1935: n. 4).
[28] See Pontow and Schubert (2006), Theorem 34, for details and proof.
[29] This strategy is not uncommon in the mathematically oriented literature; see again Pontow and Schubert (2006) for a comprehensive treatment.

Second, note that GEM is fully committed to the existence of U, a “universal entity” of which everything is part. This is not by itself a problem, barring any philosophical concerns about the gerrymendered nature of such an entity. It is, however, not without consequences. In particular, while GEM admits of models in which everything is composed of atoms as well as “gunky” models in which everything divides forever, the necessary existence of U deprives GEM of any “junky” model in which everything composes forever. Thus, while GEM admits of both atomistic and atomless extensions, adding the *Ascent* principle (P.9) would immediately result in an inconsistent theory.

5 Summary of GEM

For ease of reference, we conclude by summarizing the main axiomatizations of GEM mentioned above, with some rewriting of bound variables and dropping all redundancies:[30]

(I)	$(Pxy \wedge Pyz) \rightarrow Pxz$	*Transitivity*	(P.2)
	$PPxy \rightarrow \exists z(PPzy \wedge \neg Ozx)$	*Weak Supplementation*	(P.4)
	$\exists x\varphi x \rightarrow \exists z F_a z\varphi x$	*Unrestricted Composition*$_a$	(P.13$_a$)
	$(F_a z\varphi x \wedge Pyz) \rightarrow \exists x(\varphi x \wedge Oxy)$	*Filtration*	(P.14)
(II)	$(Pxy \wedge Pyz) \rightarrow Pxz$	*Transitivity*	(P.2)
	$PPxy \rightarrow \exists z(PPzy \wedge \neg Ozx)$	*Weak Supplementation*	(P.4)
	$\exists x\varphi x \rightarrow \exists z F_b z\varphi x$	*Unrestricted Composition*$_b$	(P.13$_b$)
(III)	Pxx	*Reflexivity*	(P.1)
	$(Pxy \wedge Pyz) \rightarrow Pxz$	*Transitivity*	(P.2)
	$Pxy \wedge Pyx) \rightarrow x = y$	*Antisymmetry*	(P.3)
	$\neg Pyx \rightarrow \exists z(Pzy \wedge \neg Ozx)$	*Strong Supplementation*	(P.5)
	$\exists x\varphi x \rightarrow \exists z F_c z\varphi x$	*Unrestricted Composition*$_c$	(P.13$_c$)
(IV)	$(Pxy \wedge Pyz) \rightarrow Pxz$	*Transitivity*	(P.3)
	$\exists x\varphi x \rightarrow \exists z(z = \sigma x\varphi x)$	*Unique Unrestricted Fusion*	(P.13)

[30] See also Simons (1987) and Hovda (2009) for additional axiom sets. The elegant axiomatization in (IV) is essentially due to Tarski (1929), though the axioms are explicitly given only in the 1956 English translation