J. Appl. Prob. **54,** 970–976 (2017)
*doi:*10.1017/jpr.2017.45

STOCHASTIC COMPARISON OF PARALLEL SYSTEMS WITH HETEROGENEOUS EXPONENTIAL COMPONENTS

BIN CHENG,* *Columbia University*
JIANTIAN WANG,** *Kean University*

Abstract

In this paper we provide a sufficient condition for mean residual life ordering of parallel systems with $n \geq 3$ heterogeneous exponential components.

Keywords: Mean residual life order; reciprocal majorization; parallel system; order statistics; proportional hazard model

2010 Mathematics Subject Classification: Primary 90B25
Secondary 60E15

1. Introduction

Order statistics play an important role in reliability theory, statistical inference, life testings, and many other areas. In reliability theory, it is well known that the kth order statistic corresponds to the lifetime of a $(n-k+1)$-out-of-n system. Thus, the study of lifetimes of k-out-of-n systems is equivalent to the study of the stochastic properties of order statistics. In particular, a 1-out-of-n system corresponds to a parallel system, and its lifetime is the largest order statistic of the ones of its components.

Parallel systems are the building blocks of more complex coherent systems, and the study of the lifetime of parallel systems has attracted much attention and many interesting results have been established. For a comprehensive survey, we refer the reader to [1] and [6].

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a parallel system whose components X_i are independent exponential random variables with hazard rate λ_i, $i = 1, \ldots, n$. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be another parallel system which is independent from $\boldsymbol{X}$ and whose components Y_i are independent exponential random variables with hazard rate μ_i, $i = 1, \ldots, n$. Let $X_{n:n} = \max\{X_1, \ldots, X_n\}$ and $Y_{n:n} = \max\{Y_1, \ldots, Y_n\}$ be the lifetimes of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Pledger and Proschan [5] established that

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succ} \boldsymbol{u} \quad \Longrightarrow \quad X_{n:n} \geq_{\mathrm{st}} Y_{n:n}.$$

Misra and Misra [4] showed that

$$\boldsymbol{\lambda} \overset{\mathrm{w}}{\succ} \boldsymbol{u} \quad \Longrightarrow \quad X_{n:n} \geq_{\mathrm{rh}} Y_{n:n}.$$

For $n = 2$, Dykstra *et al.* [2] proved that

$$(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succ} (\mu_1, \mu_2) \quad \Longrightarrow \quad X_{2:2} \geq_{\mathrm{lr}} Y_{2:2}.$$

Received 14 November 2016; revision received 27 February 2017.
* Postal address: Department of Biostatistics, Mailman School of Public Health, Columbia University, 722 West 168th Street, New York, NY 10032, USA. Email address: bc2159@cumc.columbia.edu
** Postal address: School of Mathematical Sciences, Kean University, Union, NJ 07083, USA.

Zhao and Balakrishnan [9] revealed that, under the condition $\lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2$,

$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succ} (\mu_1, \mu_2) \quad \Longrightarrow \quad X_{2:2} \ge_{\mathrm{lr}} Y_{2:2}.$$

In [8], the same authors established a result on mean residual life (MRL) ordering. They showed that, under the condition $\lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2$,

$$(\lambda_1, \lambda_2) \overset{\mathrm{rm}}{\succ} (\mu_1, \mu_2) \quad \Longrightarrow \quad X_{2:2} \ge_{\mathrm{mrl}} Y_{2:2}. \tag{1.1}$$

The formal definitions of these and other orderings relevant to this paper can be found in Section 2.

Due to the complicated nature of the stochastic comparison in terms of MRL ordering for the general n situations, (1.1) has not been extended to the situation where the system has $n \ge 3$ components.

In this paper we close this gap by extending (1.1) to the case when $n \ge 3$. The main results of this paper can be stated as, under condition

$$\lambda_1 \le \mu_1, \qquad \mu_k \le \lambda_k, \qquad k = 2, \dots, n, \tag{1.2}$$

we have

$$\boldsymbol{\lambda} \overset{\mathrm{rm}}{\succ} \boldsymbol{u} \quad \Longrightarrow \quad X_{n:n} \ge_{\mathrm{mrl}} Y_{n:n}.$$

The paper is organized as follows. In Section 2 we state the notation and definitions. In Section 3 we present the proofs of the main results concerning the MRL ordering. In Section 4 we provide a brief discussion.

2. Notation and definitions

Let X be a nonnegative continuous random variable with distribution function $F_X(t)$, survival function $\bar{F}_X(t) = 1 - F_X(t)$, and density function $f_X(t)$. The hazard function and the reversed hazard function of X are defined as $\lambda_X(t) = f_X(t)/\bar{F}_X(t)$ and $r_X(t) = f_X(t)/F_X(t)$, respectively. For two nonnegative continuous random variables X and Y, we say that

- X is larger than Y in the usual stochastic order (denoted by $X \ge_{\mathrm{st}} Y$), if $\bar{F}_X(t) \ge \bar{F}_Y(t)$;
- X is larger than Y in the reversed hazard rate order (denoted by $X \ge_{\mathrm{rh}} Y$), if $r_X(t) \ge r_Y(t)$;
- X is larger than Y in the likelihood ratio order (denoted by $X \ge_{\mathrm{lr}} Y$), if the ratio $f_X(t)/f_Y(t)$ is increasing in t;
- X is larger than Y in the MRL order (denoted by $X \ge_{\mathrm{mrl}} Y$), if $\mathbb{E}(X - t \mid X > t) \ge \mathbb{E}(Y - t \mid Y > t)$.

Given two vectors $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ with ascending entries,

- vector $\boldsymbol{a}$ is said to majorize vector $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{\mathrm{m}}{\succ} \boldsymbol{b}$, if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$, and $\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i$ for $k = 1, \dots, n-1$;
- vector $\boldsymbol{a}$ is said to weakly majorize vector $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{\mathrm{w}}{\succ} \boldsymbol{b}$, if $\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i$ for $k = 1, \dots, n$;
- vector $\boldsymbol{a}$ is said to reciprocally majorize vector $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{\mathrm{rm}}{\succ} \boldsymbol{b}$, if $\sum_{i=1}^{j} 1/a_i \ge \sum_{i=1}^{j} 1/b_i$, $j = 1, \dots, n$.

In this paper, we use $A \overset{\mathrm{sgn}}{=} B$ to denote that A and B are of the same sign.

The following lemma is used in the proof of the main theorem.

Lemma 2.1. *For $k_1 > 0,\ k_2, \ldots, k_n \geq k_1$, the function*

$$H(k_1, \ldots, k_n;\ x) = \frac{1 - \prod_{i=1}^n (1 - \mathrm{e}^{-k_i x})}{x\mathrm{e}^{-k_1 x} \prod_{i=2}^n (1 - \mathrm{e}^{-k_i x})}$$

is positive and decreasing in $x > 0$.

Proof. We have

$$H(k_1, \ldots, k_n; x) = \frac{\exp((\sum_{i=1}^n k_i)x) - \prod_{i=1}^n (\exp(k_i x) - 1)}{x \prod_{i=2}^n (\exp(k_i x) - 1)}.$$

Let $u_i = \mathrm{e}^{k_i x} - 1$. Then

$$H(k_1, \ldots, k_n; x) = \frac{\prod_{i=1}^n (u_i + 1) - \prod_{i=1}^n u_i}{x \prod_{i=2}^n u_i} = \frac{1}{x u_2 \cdots u_n} + \sum_{l=1}^{n-1} \sum_{i_1 < i_2 < \cdots < i_l} \frac{u_{i_1} u_{i_2} \cdots u_{i_l}}{x u_2 \cdots u_n}.$$

Since each term in the above summation is a positive and decreasing function of x, the lemma is thus proved. □

3. Proofs of the theorems

Denote $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. Without loss of generality, we assume that $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$. Define $F(\boldsymbol{\lambda};\ t) = \prod_{i=1}^n (1 - \mathrm{e}^{-\lambda_i t})$. The MRL function $X_{n:n}$ is

$$\varphi_{\boldsymbol{\lambda}}(t) = \frac{\int_t^\infty \{1 - F(\boldsymbol{\lambda};\ u)\}\, \mathrm{d}u}{1 - F(\boldsymbol{\lambda};\ t)}.$$

Similarly, the MRL function of $Y_{n:n}$ is $\varphi_{\boldsymbol{\mu}}(t)$.

For $0 < x_1 \leq \cdots \leq x_n$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$, consider the function

$$\Phi(\boldsymbol{x}) = \frac{\int_1^\infty \{1 - F(\boldsymbol{x};\ u)\}\, \mathrm{d}u}{1 - F(\boldsymbol{x};\ 1)}.$$

Since $X_{n:n} >_{\mathrm{mrl}} Y_{n:n}$ is equivalent to $\Phi(\boldsymbol{\lambda} t) \geq \Phi(\boldsymbol{\mu} t)$, it is enough to study the monotonicity of $\Phi(\boldsymbol{x})$ along certain vector fields. We first establish a monotonicity result.

Theorem 3.1. *Let $\pi_i \geq 0,\ i = 2, \ldots, n$, such that $\sum_{i=2}^n \pi_i \leq 1$. Then*

$$\nabla \Phi_{(x_1^2, -\pi_2 x_2^2, \ldots, -\pi_n x_n^2)}(\boldsymbol{x}) \leq 0.$$

Proof. First, we have

$$\frac{\partial F(\boldsymbol{x};\ u)}{\partial x_i} = \frac{u\mathrm{e}^{-x_i u}}{1 - \mathrm{e}^{-x_i u}} F(\boldsymbol{x};\ u).$$

Therefore,

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_i} &\overset{\mathrm{sgn}}{=} \int_1^\infty \left\{ F(\boldsymbol{x};\ 1) \frac{\mathrm{e}^{-x_i}}{1 - \mathrm{e}^{-x_i}} [1 - F(\boldsymbol{x};\ t)] - F(\boldsymbol{x};\ t) \frac{t\mathrm{e}^{-x_i t}}{1 - \mathrm{e}^{-x_i t}} [1 - F(\boldsymbol{x};1)] \right\} \mathrm{d}t \\
&= \frac{1}{x_i^2} \int_{x_i}^\infty K_i(u)\, \mathrm{d}u,
\end{aligned}$$

where $K_i(u) = F(\boldsymbol{x};\ 1)\eta(x_i)[1 - F(\boldsymbol{x};\ u/x_i)] - F(\boldsymbol{x};\ u/x_i)\eta(u)[1 - F(\boldsymbol{x};\ 1)]$, and $\eta(x) = x\mathrm{e}^{-x}/(1 - \mathrm{e}^{-x})$. For simplicity, we write $F(u)$ for $F(\boldsymbol{x};\ u)$ for the rest of the paper.

By Lemma 2.1, when $u \geq x_1$,

$$
\begin{aligned}
K_1(u) &\overset{\text{sgn}}{=} \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_1)} \\
&= H\left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1};\ u\right) - H\left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1};\ x_1\right) \\
&\leq 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\nabla\Phi_{(x_1^2, -\pi_2 x_2^2, \dots, -\pi_n x_n^2))} &\overset{\text{sgn}}{=} \nabla\Phi_{[(1-\sum_{i=2}^n \pi_i)x_1^2, 0, \dots, 0]} + \sum_{i=2}^n \nabla\Phi_{(\pi_i x_1^2, 0, \dots, 0, -\pi_i x_i^2, 0, \dots, 0)} \\
&= \left(1 - \sum_{i=2}^n \pi_i\right) x_1^2 \frac{\partial\Phi}{\partial x_1} + \sum_{i=2}^n \pi_i \left\{ x_1^2 \frac{\partial\Phi}{\partial x_1} - x_i^2 \frac{\partial\Phi}{\partial x_i} \right\} \\
&= \left(1 - \sum_{i=2}^n \pi_i\right) \int_{x_1}^\infty K_1(u)\,\mathrm{d}u \\
&\quad + \sum_{i=2}^n \pi_i \left\{ \int_{x_1}^\infty K_1(u)\,\mathrm{d}u - \int_{x_i}^\infty K_i(u)\,\mathrm{d}u \right\} \\
&\leq \sum_{i=2}^n \pi_i \int_{x_i}^\infty [K_1(u) - K_i(u)]\,\mathrm{d}u \\
&\leq \sum_{i=2}^n \pi_i \int_{u \geq x_i,\, K_i(u) \leq 0} [K_1(u) - K_i(u)]\,\mathrm{d}u.
\end{aligned}
$$

Assume that $K_i(u) \leq 0$. Since $F(u)$ is positive and increasing, we have

$$
\begin{aligned}
K_i(u) - K_1(u) &= \left\{ \frac{1-F(u/x_i)}{F(u/x_i)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_i)} \right\} F(1)\eta(x_i) F\left(\frac{u}{x_i}\right)\eta(u) \\
&\quad - \left\{ \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_1)} \right\} F(1)\eta(x_1) F\left(\frac{u}{x_1}\right)\eta(u) \\
&\geq \left\{ \frac{1-F(u/x_i)}{F(u/x_i)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_i)} \right\} F(1)\eta(x_i) F\left(\frac{u}{x_1}\right)\eta(u) \\
&\quad - \left\{ \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_1)} \right\} F(1)\eta(x_1) F\left(\frac{u}{x_1}\right)\eta(u) \\
&\overset{\text{sgn}}{=} \left\{ \frac{1-F(u/x_i)}{F(u/x_i)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_i)} \right\} \eta(x_i) \\
&\quad - \left\{ \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_1)} \right\} \eta(x_1) \\
&= \frac{1-F(u/x_i)}{F(u/x_i)\eta(u)}\eta(x_i) - \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)}\eta(x_1) \\
&= \frac{1-F(u/x_i)}{F(u/x_i)\eta(x_1 u/x_i)} \frac{\eta(x_1 u/x_i)\eta(x_i)}{\eta(u)} - \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)}\eta(x_1)
\end{aligned}
$$

$$\geq \left\{ H\left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}; \frac{x_1}{x_i}u\right) - H\left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}; u\right)\right\}\eta(x_1)$$
$$\geq 0,$$

where the second to last inequality is because function $\eta(x_1u/x_i)/\eta(u)$ is increasing over $u \geq x_i$, hence, $\eta(x_1u/x_i)/\eta(u) \geq \eta(x_1)/\eta(x_i)$, or, equivalently, $\eta(x_1u/x_i)\eta(x_i)/\eta(u) \geq \eta(x_1)$, and the last inequality is by Lemma 2.1. Therefore,

$$\nabla\Phi_{(x_1^2, -\pi_2 x_2^2, \dots, -\pi_n x_n^2)} \leq \sum_{i=2}^{n} \pi_i \int_{u \geq x_i,\, K_i(u) \leq 0} [K_1(u) - K_i(u)]\, \mathrm{d}u \leq 0. \qquad \square$$

Define

$$\Omega_{\mathrm{rm}}(\boldsymbol{\lambda}) = \{\boldsymbol{\mu} \colon \boldsymbol{\mu} \overset{\mathrm{rm}}{\prec} \boldsymbol{\lambda},\, \mu_k \leq \lambda_k,\, k \geq 2\}.$$

Theorem 3.2. *Denote by $X_{n:n}(\boldsymbol{\lambda})$ the parallel system with hazard rate parameter $\boldsymbol{\lambda}$. For any $\boldsymbol{\mu} \in \Omega_{\mathrm{rm}}(\boldsymbol{\lambda})$, let $Y_{n:n}(\boldsymbol{\mu})$ be another independent parallel system with hazard rate parameter $\boldsymbol{\mu}$. We have*

$$X_{n:n}(\boldsymbol{\lambda}) >_{\mathrm{mrl}} Y_{n:n}(\boldsymbol{\mu}).$$

Proof. For any $\boldsymbol{\mu} \in \Omega_{\mathrm{rm}}(\boldsymbol{\lambda})$, let k_0 be the smallest k such that

$$\sum_{i=1}^{k} \frac{1}{\mu_i} = \sum_{i=1}^{k} \frac{1}{\lambda_i} \tag{3.1}$$

holds. It is easy to see that under condition $\mu_k \leq \lambda_k,\ k \geq 2$, for any $k \geq k_0$, we should also have

$$\sum_{i=1}^{k} \frac{1}{\mu_i} = \sum_{i=1}^{k} \frac{1}{\lambda_i}, \qquad k \geq k_0,$$

which also implies that $\mu_k = \lambda_k,\ k \geq k_0 + 1$.

If $k_0 = 1$ then we have $\mu_1 = \lambda_1$, condition $\boldsymbol{\mu} \overset{\mathrm{rm}}{\prec} \boldsymbol{\lambda}$ and (1.2) imply that $\boldsymbol{\mu} = \boldsymbol{\lambda}$, thus, the conclusion holds trivially. Now we assume that k_0 exists and $k_0 \geq 2$. Then we know that point

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_{k_0}, \mu_{k_0+1}, \dots, \mu_n) = (\mu_1, \dots, \mu_{k_0}, \lambda_{k_0+1}, \dots, \lambda_n)$$

is on the $(k_0 - 1)$-dimensional manifold

$$\sum_{i=1}^{k_0} \frac{1}{x_i} = \sum_{i=1}^{k_0} \frac{1}{\lambda_i}, \qquad x_j \equiv \lambda_j,\ j = k_0 + 1, \dots, n. \tag{3.2}$$

Consider the vector field $\boldsymbol{a}(\boldsymbol{x}) = (x_1^2, -\pi_2 x_2^2, \dots, -\pi_{k_0} x_{k_0}^2, 0, \dots, 0)$, where $\sum_{i=2}^{k_0} \pi_i = 1$. The outer normal vector of manifold (3.2) at point $\boldsymbol{x}$ is $(-1/x_1^2, \dots, -1/x_n^2)$, which is orthogonal to $\boldsymbol{a}(\boldsymbol{x})$. This means that the vector $\boldsymbol{a}(\boldsymbol{x})$ is tangent to the manifold (3.2) at point $\boldsymbol{x}$. Therefore, we can start from $\boldsymbol{\lambda}$ and travel on this manifold along a tangent vector field $\boldsymbol{a}(\boldsymbol{x})$ to reach $\boldsymbol{\mu}$. In fact, set

$$\left\{\boldsymbol{\lambda}(t) = \left(\frac{\lambda_1\mu_1}{(1-t)\mu_1 + t\lambda_1}, \dots, \frac{\lambda_{k_0}\mu_{k_0}}{(1-t)\mu_{k_0} + t\lambda_{k_0}}, \lambda_{k_0+1}, \dots, \lambda_n\right);\ 0 \leq t \leq 1\right\}$$

indexed by t is a curve on manifold (3.2) which connects points $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. To see this,

$$\sum_{i=1}^{k_0} \frac{1}{(\lambda_i\mu_i)/[(1-t)\mu_i + t\lambda_i]} + \sum_{j=k_0+1}^{n} \frac{1}{\lambda_j} = (1-t)\bigg(\sum_{i=1}^{k_0} \frac{1}{\lambda_i}\bigg) + t\bigg(\sum_{i=1}^{k_0} \frac{1}{\mu_i}\bigg) + \sum_{j=k_0+1}^{n} \frac{1}{\lambda_j}$$
$$= \sum_{i=1}^{n} \frac{1}{\lambda_i},$$

where the last equality is by (3.1). We show that the tangent vector field of curve $\boldsymbol{\lambda}(t)$ at t, denoted by $\boldsymbol{\lambda}'(t)$, is of the form required in Theorem 3.1. In fact, denote $\lambda_i(t) = \lambda_i\mu_i/[(1-t)\mu_i + t\lambda_i]$, $i = 1, \dots, k_0$, then

$$\boldsymbol{\lambda}'(t) = \bigg(\bigg(\frac{1}{\lambda_1} - \frac{1}{\mu_1}\bigg)(\lambda_1(t))^2, \dots, \bigg(\frac{1}{\lambda_{k_0}} - \frac{1}{\mu_{k_0}}\bigg)(\lambda_{k_0}(t))^2, 0, \dots, 0\bigg)$$
$$\overset{\text{sgn}}{=} \bigg(1 \cdot \lambda_1^2(t), -\frac{1/\mu_2 - 1/\lambda_2}{1/\lambda_1 - 1/\mu_1}\lambda_2^2(t), \dots, -\frac{1/\mu_{k_0} - 1/\lambda_{k_0}}{1/\lambda_1 - 1/\mu_1}\lambda_{k_0}^2(t), 0, \dots, 0\bigg),$$

which satisfies the form required by Theorem 3.1 with

$$\pi_i = \frac{1/\mu_i - 1/\lambda_i}{1/\lambda_1 - 1/\mu_1} \ge 0 \quad \text{for } i \ge 2$$

because

$$\sum_{i=2}^{n} \pi_i = \sum_{i=2}^{n} \frac{1/\mu_i - 1/\lambda_i}{1/\lambda_1 - 1/\mu_1} = \sum_{i=2}^{k_0} \frac{1/\mu_i - 1/\lambda_i}{1/\lambda_1 - 1/\mu_1}$$
$$= \frac{1}{1/\lambda_1 - 1/\mu_1}\bigg(\sum_{i=2}^{n} \frac{1}{\mu_i} - \sum_{i=2}^{n} \frac{1}{\lambda_i}\bigg)$$
$$= \frac{1}{1/\lambda_1 - 1/\mu_1}\bigg(\frac{1}{\lambda_1} - \frac{1}{\mu_i}\bigg)$$
$$= 1,$$

where the second to last equality is by condition (3.1) for $k = k_0$. Therefore, by Theorem 3.1, $\Phi(\boldsymbol{\lambda}) \ge \Phi(\boldsymbol{\mu})$, or $X_{n:n}(\boldsymbol{\lambda}) >_{\text{mrl}} X_{n:n}(\boldsymbol{\mu})$.

If such a k_0 does not exist, that means that

$$\sum_{i=1}^{n} \frac{1}{\mu_i} < \sum_{i=1}^{n} \frac{1}{\lambda_i}. \tag{3.3}$$

Choose μ_1^* such that

$$\frac{1}{\mu_1^*} + \sum_{i=2}^{n} \frac{1}{\mu_i} = \frac{1}{\lambda_1} + \sum_{i=2}^{n} \frac{1}{\lambda_i}. \tag{3.4}$$

By assumption (1.2), $\mu_k \le \lambda_k$, $k \ge 2$, thus,

$$\sum_{i=2}^{n} \frac{1}{\mu_i} \ge \sum_{i=2}^{n} \frac{1}{\lambda_i}. \tag{3.5}$$

In view of (3.3)–(3.5), we have $\mu_1 > \mu_1^* \geq \lambda_1$. Point $\boldsymbol{\mu}^* = (\mu_1^*, \mu_2, \ldots, \mu_n)$ is on manifold

$$\sum_{i=1}^{n} \frac{1}{x_i} = \sum_{i=1}^{n} \frac{1}{\lambda_i}, \qquad \lambda_1 \leq \mu_1^*.$$

Therefore, we have shown that we can start from $\boldsymbol{\lambda}$ and travel along a vector field $\boldsymbol{a}(\boldsymbol{x})$ on this manifold to reach $\boldsymbol{\mu}^*$, and then along vector (1,0, …, 0) to reach $\boldsymbol{\mu}$. Thus,

$$\Phi(\boldsymbol{\lambda}) \geq \Phi(\boldsymbol{\mu}^*) \geq \Phi(\boldsymbol{\mu}),$$

which implies that $X_{n:n}(\boldsymbol{\lambda}) >_{\text{mrl}} X_{n:n}(\boldsymbol{\mu})$. □

4. Discussion

First, in proving Theorem 3.2, condition (1.2) seems crucial. Note that when $n = 2$, condition (1.2) becomes $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$, which is the condition imposed by [3], [8], and [9]. Therefore, condition (1.2) can be considered as a high-dimensional analogy of $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$.

Second, we believe the following inequality holds:

$$\nabla \Phi_{(x_1^2, \pm\pi_2 x_2^2, \ldots, \pm\pi_n x_n^2)}(\boldsymbol{x}) \leq 0$$

in Theorem 3.1. If the above conjecture is correct then condition (1.2) can be weakened.

Finally, similar to [7], the result in this paper can be readily extended to parallel systems whose components follow proportional hazard models. That is, the survival function of X_i satisfies $\bar{F}_{X_i}(t) = \mathrm{e}^{\lambda_i R(t)}$, where $R(t)$ is a baseline cumulative hazard function.

Acknowledgements

The authors would like to thank the editor, the associate editor, and an anonymous referee whose comments lead to an improved presentation.

References

[1] Balakrishnan, N. and Zhao, P. (2013). Ordering properties of order statistics from heterogeneous populations: a review with an emphasis on some recent developments. *Prob. Eng. Inf. Sci.* **27,** 403–443.

[2] Dykstra, R., Kochar, S. and Rojo, J. (1997). Stochastic comparisons of parallel systems of heterogeneous exponential components. *J. Statist. Planning Infer.* **65,** 203–211.

[3] Joo, S. and Mi, J. (2010). Some properties of hazard rate functions of systems with two components. *J. Statist. Planning Infer.* **140,** 444–453.

[4] Misra, N., and Misra, A. K. (2013). On comparison of reversed hazard rates of two parallel systems comprising of independent gamma components. *Statist. Prob. Lett.* **83**, 1567–1570.

[5] Pledger, G. and Proschan, F. (1971). Comparison of order statistics and of spacings from heterogeneous distributions. In *Optimizing Methods in Statistics*, Academic Press, New York, pp. 89–113.

[6] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer, New York.

[7] Torrado, N. and Kochar, S. C. (2015). Stochastic order relations among parallel systems from Weibull distributions. *J. Appl. Prob.* **52,** 102–116.

[8] Zhao, P. and Balakrishnan, N. (2011). MRL ordering of parallel systems with two heterogeneous components. *J. Statist. Planning Infer.* **141,** 631–638.

[9] Zhao, P. and Balakrishnan, N. (2011). Some characterization results for parallel systems with two heterogeneous exponential components. *Statistics* **45,** 593–604.