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Likelihood ratio ordering for parallel systems with exponential components

Jiantian Wang[a] and Bin Cheng[b]

[a]School of Mathematical Sciences, Kean University, Union, NJ, USA; [b]Department of Biostatistics, Mailman School of Public Health, Columbia University, New York, NY, USA

ABSTRACT
In this paper, we introduce the concept of *l*-order and conjecture that the *l*-order of hazard rate vectors of components implies the likelihood ratio order of parallel systems. We prove this conjecture when the number of components is no more than 5.

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1. Introduction

The research on stochastic comparison of lifetimes of parallel systems has a long history, and many interesting results have been established. See, for instance, Proschan and Sethuraman (1976), Dykstra, Kochar, and Rojo (1997), Khaledi and Kochar (2000), Kochar and Xu (2007), Kochar and Xu (2011), Zhao and Balakrishnan (2011), Yan, Da, and Zhao (2013), Torrado and Lillo (2013), Torrado and Kochar (2015), and a survey article of Balakrishnan and Zhao (2013).

The stochastic comparison of lifetimes of parallel systems with multiple components is technically challenging. In fact, this issue has not been well investigated even for the simplest case of exponential components. In this paper, we study the likelihood ratio ordering of parallel systems with exponential components.

For a parallel system consisting of n components whose lifetimes are $X_i \sim \exp(\lambda_i)$, where λ_i is the hazard rate of ith component, $i = 1, \cdots, n$, its lifetime is defined as $X_{n:n} = \max\{X_1, \cdots, X_n\} = T(\lambda_1, \cdots, \lambda_n)$. By symmetry, in the sequel, we assume $\lambda_1 \le \cdots \le \lambda_n$. Similar notation for $Y_{n:n} = \max\{Y_1, \cdots, Y_n\} = T(\mu_1, \cdots, \mu_n)$.

Dykstra, Kochar, and Rojo (1997) showed that

$$(\lambda_1, \lambda_2) \overset{m}{\succ} (\mu_1, \mu_2) \Rightarrow T(\lambda_1, \lambda_2) \ge_{lr} T(\mu_1, \mu_2),$$

and Boland, El-Neweihi, and Proschan (1994) constructed an example showing that

CONTACT Jiantian Wang jwang@kean.edu School of Mathematical Sciences, Kean University, Union, NJ 07083, USA.
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$$(\lambda_1, \cdots, \lambda_n) \overset{m}{\succ} (\mu_1, \cdots, \mu_n) \nRightarrow T(\lambda_1, \cdots, \lambda_n) \geq_{lr} T(\mu_1, \cdots, \mu_n),$$

where $\overset{m}{\succ}$ stands for the majorization order, and $\geq_{lr}$ for the likelihood ratio order. Since then, there has been a long lasting interest about what kind of order of hazard rate vectors of components can guarantee $T(\lambda_1, \cdots, \lambda_n) \geq_{lr} T(\mu_1, \cdots, \mu_n)$.

Some results on special cases have been established. For instance, when $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$, Mao and Hu (2010) showed that when $\mu \geq \bar{\lambda}$, where $\bar{\lambda}$ is the arithmetic average of $\lambda_1, \cdots, \lambda_n, T(\lambda_1, \cdots, \lambda_n) \geq_{lr} T(\mu_1, \cdots, \mu_n)$.

It is well known that when $\lambda_1 = \mu_1$ there is no likelihood ratio order between $T(\lambda_1, \cdots, \lambda_n)$ and $T(\mu_1, \cdots, \mu_n)$. Hence, a necessary condition for $T(\lambda_1, \cdots, \lambda_n) \geq_{lr} T(\mu_1, \cdots, \mu_n)$ is $\lambda_1 < \mu_1$.

For lifetime $X \sim \exp(\lambda)$, smaller λ implies longer lifetime. Consider a parallel system with four components with hazard rate vector $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. If the hazard rate vector changes in the direction $(1, 1, 1, 1)$, which indicates that each component's lifetime gets shorter, we can expect the lifetime of the system also gets shorter. If the hazard rate vector changes in the direction $(1, 1, -1, -1)$, which indicates the best two components become worse while the other two become better, we can expect the lifetime of the system to be shorter. If the hazard rate vector changes in the direction (1, $-$1, $-$1, $-$1), indicating that the best component becomes worse while the others become better, it is unclear how the lifetime of the system changes.

Inspired by these observations, we introduce the concept of l-order for the hazard rate vectors. A vector $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$ with $v_1 = 1$ is called a basic l-vector if there exists an integer p, $1 \leq p \leq n$, such that, for $i \leq p$, v_i is 1 or 0 while for $i > p$, v_i is -1 or 0, and the number of 1's is no less than that of -1's. For instance, $(1, 0, 1, 0, -1)$ and $(1, 1, 1, -1, -1)$ are basic l-vectors, but $(1, -1, 0, -1)$ and $(1, 1, -1, -1, -1)$ are not.

We say vector $\boldsymbol{u}$ is a positive linear combination of vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$, if $\boldsymbol{u} = a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m$, with $a_i \geq 0$. Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be equal, if $\boldsymbol{a} = k\boldsymbol{b}$ for some $k > 0$.

A vector is called an l-vector if it is a positive linear combination of basic l-vectors. For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we say $\boldsymbol{a}$ is l-larger than $\boldsymbol{b}$, denoted as $\boldsymbol{a} \overset{l}{\succ} \boldsymbol{b}$, if $\boldsymbol{b} - \boldsymbol{a}$ is an l-vector.

In the case of $n = 2$, $(\lambda_1, \lambda_2) \overset{m}{\succ} (\mu_1, \mu_2)$ is equivalent to $(\mu_1, \mu_2) - (\lambda_1, \lambda_2) = (1, -1)$. Dykstra, Kochar, and Rojo (1997) showed that

$$(\lambda_1, \lambda_2) \overset{m}{\succ} (\mu_1, \mu_2) \Rightarrow T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2).$$

Zhao and Balakrishnan (2011) established that when $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$ and $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2)$. Yan, Da, and Zhao (2013) showed that, when $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2$ and $\lambda_1 + \mu_2 \leq \mu_1 + \lambda_2, T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2)$. Noticing that for $n = 2$, the basic l-vectors are $(1, -1)$, (1, 0), and (1, 1), the three results mentioned above can be consolidated as

$$(\lambda_1, \lambda_2) \overset{l}{\succ} (\mu_1, \mu_2) \Rightarrow T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2). \tag{1}$$

This motivates us to conjecture that the above result holds for general n. That is,

$$(\lambda_1, \cdots, \lambda_n) \overset{l}{\succ} (\mu_1, \cdots, \mu_n) \Rightarrow T(\lambda_1, \cdots, \lambda_n) \geq_{lr} T(\mu_1, \cdots, \mu_n). \tag{2}$$

The main objective of this paper is to prove this conjecture for $n \leq 5$. Our computer simulations show that the conjecture should hold for $n \leq 10$. But a proof for general n is not currently available.

The paper is organized as follows. Section 2 provides the proof of the main result. Section 3 is a short discussion. The proofs of the auxiliary results are relegated to the Appendix.

2. Main result and proof

Theorem 2.1. *For $n \leq 5$,*

$$(\lambda_1, \cdots, \lambda_n) \overset{l}{\succ} (\mu_1, \cdots, \mu_n) \Rightarrow T(\lambda_1, \cdots, \lambda_n) \geq_{lr} T(\mu_1, \cdots, \mu_n).$$

Proof. Let $X = T(\lambda_1, \cdots, \lambda_n), Y = T(\mu_1, \cdots, \mu_n)$. The ratio of the reversed hazard rate function of X over that of Y is

$$\psi(t) = \frac{r_{\boldsymbol{\lambda}}(t)}{r_{\boldsymbol{\mu}}(t)} = \frac{\sum_{i=1}^n \frac{\lambda_i e^{-\lambda_i t}}{1-e^{-\lambda_i t}}}{\sum_{i=1}^n \frac{\mu_i e^{-\mu_i t}}{1-e^{-\mu_i t}}} \overset{\Delta}{=} \frac{\varphi(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\mu}; t)},$$

where $A \overset{\Delta}{=} B$ means A is denoted as B. By Theorem 1.C.4 of Shaked and Shanthikumar (2007), to show $X \geq_{lr} Y$, we just need to show $\psi'(t) \geq 0$.

For convenience, we denote $A \overset{\text{sgn}}{=} B$ if the signs of A and B are the same. We have,

$$\begin{aligned}\psi'(t) &\overset{\text{sgn}}{=} \varphi_t'(\boldsymbol{\lambda}; t)\varphi(\boldsymbol{\mu}; t) - \varphi(\boldsymbol{\lambda}; t)\varphi_t'(\boldsymbol{\mu}; t) \\ &\overset{\text{sgn}}{=} \frac{\varphi_t'(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\lambda}; t)} - \frac{\varphi_t'(\boldsymbol{\mu}; t)}{\varphi(\boldsymbol{\mu}; t)},\end{aligned}$$

where

$$\frac{\varphi_t'(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\lambda}; t)} = -\frac{\sum_{i=1}^n \frac{\lambda_i^2 e^{-\lambda_i t}}{(1-e^{-\lambda_i t})^2}}{\sum_{i=1}^n \frac{\lambda_i e^{-\lambda_i t}}{1-e^{-\lambda_i t}}} = -\frac{\sum_{i=1}^n \frac{\lambda_i^2 e^{\lambda_i t}}{(e^{\lambda_i t}-1)^2}}{\sum_{i=1}^n \frac{\lambda_i}{e^{\lambda_i t}-1}},$$

and similarly for $\frac{\varphi_t'(\boldsymbol{\mu}; t)}{\varphi(\boldsymbol{\mu}; t)}$.

Let

$$\Phi(x_1, \cdots, x_n) = \frac{\sum_{i=1}^n b(x_i) d(x_i)}{\sum_{i=1}^n b(x_i)}, \qquad 0 < x_1 \leq x_2 \leq \cdots \leq x_n.$$

where $b(x) = x/(e^x - 1), d(x) = xe^x/(e^x - 1)$. Since

$$\frac{\varphi_t'(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\lambda}; t)} = -t^{-1}\Phi(\lambda_1 t, \cdots, \lambda_n t),$$

we conclude that $\psi'(t) \geq 0$ is equivalent to

$$\Phi(\mu_1 t, \cdots, \mu_n t) - \Phi(\lambda_1 t, \cdots, \lambda_n t) \geq 0$$

for any $t > 0$. Therefore, to prove Theorem 2.1, it suffices to demonstrate that function $\Phi(x_1, \cdots, x_n)$ is increasing in l-vector directions.

Let $\boldsymbol{u} = (1, u_2, \cdots, u_n)$ be a given basic l-vector. We first consider the case where u_i is either 0 or 1, $i = 2, \cdots, n$. It is easy to see that such an l-vector can be written as a positive linear combination of $\boldsymbol{\delta}_1 = (1, 0, \cdots, 0)$ and $\boldsymbol{\delta}_k$, whose first element is 1, kth element is 4, and all others are zero, $k = 2, \cdots, n$. For instance, when $n = 5$,

$$(1,1,1,1,1) \overset{\text{sgn}}{=} \sum_{k=2}^{5} \boldsymbol{\delta}_k, \qquad (1,1,0,0,0) \overset{\text{sgn}}{=} 3\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2.$$

Thus, we only need to show that the function Φ increases in the vectors $\boldsymbol{\delta}_i, i = 1, \cdots, n$.

Calculating

$$\frac{\partial \Phi}{\partial x_i} \overset{\text{sgn}}{=} \left[b'(x_i)d(x_i) + b(x_i)d'(x_i)\right] \sum_{j=1}^{n} b(x_j) - b'(x_i) \sum_{j=1}^{n} b(x_j)d(x_j),$$

and noticing that $d'(x) \geq 0$ and $b'(x) \leq 0$, we have

$$\begin{aligned}
\nabla_{\boldsymbol{\delta}_1} \Phi \overset{\text{sgn}}{=} \frac{\partial \Phi}{\partial x_1} &= \left[b'(x_1)d(x_1) + b(x_1)d'(x_1)\right]\sum_{j=1}^{n} b(x_j) - b'(x_1)\sum_{j=1}^{n} b(x_j)d(x_j) \\
&\geq b'(x_1)d(x_1)\sum_{j=1}^{n} b(x_j) - b'(x_1)\sum_{j=1}^{n} b(x_j)d(x_j) \\
&= b'(x_1)\sum_{j=1}^{n} b(x_j)\left[d(x_1) - d(x_j)\right] \geq 0.
\end{aligned}$$

In the direction $\boldsymbol{\delta}_k, k = 2, \cdots, n,$ we have

$$\begin{aligned}
\nabla_{\boldsymbol{\delta}_k} \Phi &\overset{\text{sgn}}{=} \frac{\partial \Phi}{\partial x_1} + 4\frac{\partial \Phi}{\partial x_k} \\
&= \left[b'(x_1)d(x_1) + b(x_1)d'(x_1)\right]\sum_{j=1}^{n} b(x_j) - b'(x_1)\sum_{j=1}^{n} b(x_j)d(x_j) \\
&\quad + 4\left[b'(x_k)d(x_k) + b(x_k)d'(x_k)\right]\sum_{j=1}^{n} b(x_j) - 4b'(x_k)\sum_{j=1}^{n} b(x_j)d(x_j) \\
&= \sum_{j=1}^{n} I(j)b(x_j),
\end{aligned}$$

where

$$I(j) = b'(x_1)d(x_1) + b(x_1)d'(x_1) + 4b'(x_k)d(x_k) + 4b(x_k)d'(x_k) - \left[b'(x_1) + 4b'(x_k)\right]d(x_j).$$

Since $d(x)$ is an increasing function and $b'(x) \leq 0$, $I(j)$ increases with j. We have,

$$\begin{aligned}
I(1) &= b'(x_1)d(x_1) + b(x_1)d'(x_1) + 4b'(x_k)d(x_k) + 4b(x_k)d'(x_k) - \left[b'(x_1) + 4b'(x_k)\right]d(x_1) \\
&= b(x_1)d'(x_1) + 4b(x_k)d'(x_k) + 4b'(x_k)[d(x_k) - d(x_1)].
\end{aligned}$$

From Lemma A.2 with $(\alpha, \beta) = (4, 4)$, we know $I(1) \geq 0$, and hence, $\nabla_{\boldsymbol{\delta}_k} \Phi \geq 0$.

Now we consider the case when the basic l-vector contains components of -1. When $n \leq 5$, after suppressing the zero elements and renaming the subscripts, a basic l-vector must be one of the following 6 vectors: $\boldsymbol{v}_1 = (1, -1), \boldsymbol{v}_2 = (1, 1, -1), \boldsymbol{v}_3 = (1, 1, 1, -1)$, $\boldsymbol{v}_4 = (1, 1, 1, 1, -1), \boldsymbol{v}_5 = (1, 1, -1, -1)$, and $\boldsymbol{v}_6 = (1, 1, 1, -1, -1)$.

Let $\boldsymbol{v} = (1, v_2, \cdots, v_n)$ be a basic l-vector with some elements of -1. Write

$$
\begin{aligned}
\nabla_v \Phi &= \sum_{i=1}^{n} v_i \frac{\partial \Phi}{\partial x_i} \\
&\overset{\text{sgn}}{=} \sum_{i=1}^{n} v_i \left[b'(x_i) d(x_i) + b(x_i) d'(x_i) \right] \sum_{j=1}^{n} b(x_j) - \sum_{i=1}^{n} v_i b'(x_i) \sum_{j=1}^{n} b(x_j) d(x_j) \\
&= \sum_{j=1}^{n} R(j) b(x_j),
\end{aligned}
$$

where

$$
\begin{aligned}
R(j) &= \sum_{i=1}^{n} v_i \left[b'(x_i) d(x_i) + b(x_i) d'(x_i) \right] - \sum_{i=1}^{n} v_i b'(x_i) d(x_j) \\
&= \sum_{i=1}^{n} v_i b'(x_i) \left[d(x_i) - d(x_j) \right] + \sum_{i=1}^{n} v_i b(x_i) d'(x_i).
\end{aligned}
$$

Since $-b'(x)$ is decreasing, we can see that for any of the 6 above-mentioned basic l-vectors, $-\sum_{i=1}^{n} v_i b'(x_i)$ is positive. Noticing that $d(x)$ is increasing, so, $R(j)$ is increases with j. Thus, to prove $\nabla_{\boldsymbol{v}} \Phi \geq 0$, it suffices to show $R(1) \geq 0$.

For $\boldsymbol{v} = \boldsymbol{v}_1$, since $b(x)d'(x)$ and $b(x)$ are decreasing,

$$
\begin{aligned}
R(1) &= \sum_{i=1}^{n} v_i b'(x_i) [d(x_i) - d(x_1)] + \sum_{i=1}^{n} v_i b(x_i) d'(x_i) \\
&= b(x_1) d'(x_1) - b(x_2) d'(x_2) - b'(x_2) [d(x_2) - d(x_1)] \\
&\geq 0.
\end{aligned}
$$

For $\boldsymbol{v} = \boldsymbol{v}_2$,

$$
\begin{aligned}
R(1) &= \sum_{i=1}^{n} v_i b'(x_i) [d(x_i) - d(x_1)] + \sum_{i=1}^{n} v_i b(x_i) d'(x_i) \\
&= b(x_1) d'(x_1) + b(x_2) d'(x_2) - b(x_3) d'(x_3) \\
&\quad + b'(x_2) [d(x_2) - d(x_1)] - b'(x_3) [d(x_3) - d(x_1)] \\
&\geq b(x_1) d'(x_1) + b'(x_2) [d(x_2) - d(x_1)] \geq 0,
\end{aligned}
$$

where the last inequality is by Lemma A.2 with $(\alpha, \beta) = (0, 1)$.

For $\boldsymbol{v} = \boldsymbol{v}_3$,

$$\begin{aligned}R(1) &= \sum_{i=1}^{n} v_i b'(x_i)[d(x_i) - d(x_1)] + \sum_{i=1}^{n} v_i b(x_i) d'(x_i)\\ &= b(x_1)d'(x_1) + b(x_2)d'(x_2) + b(x_3)d'(x_3) - b(x_4)d'(x_4)\\ &\quad + b'(x_2)[d(x_2) - d(x_1)] + b'(x_3)[d(x_3) - d(x_1)] - b'(x_4)[d(x_4) - d(x_1)]\\ &\geq b(x_1)d'(x_1) + b(x_2)d'(x_2) + b'(x_2)[d(x_2) - d(x_1)] + b'(x_3)[d(x_3) - d(x_1)]\\ &\geq \frac{1}{2}\{b(x_1)d'(x_1) + b(x_2)d'(x_2) + 2b'(x_2)[d(x_2) - d(x_1)]\}\\ &+ \frac{1}{2}\{b(x_1)d'(x_1) + b(x_3)d'(x_3) + 2b'(x_3)[d(x_3) - d(x_1)]\}.\end{aligned}$$

By Lemma A.2 with $(\alpha, \beta) = (1, 2)$, we obtain $R(1) \geq 0$.

For $\boldsymbol{v} = \boldsymbol{v}_4$,

$$\begin{aligned}R(1) &= \sum_{i=1}^{n} v_i b'(x_i)[d(x_i) - d(x_1)] + \sum_{i=1}^{n} v_i b(x_i) d'(x_i)\\ &= \sum_{i=1}^{4} b(x_i)d'(x_i) + \sum_{i=2}^{4} b'(x_i)[d(x_i) - d(x_1)] - b(x_5)d'(x_5) - b'(x_5)[d(x_5) - d(x_1)]\\ &\geq \sum_{i=1}^{3} b(x_i)d'(x_i) + \sum_{i=2}^{4} b'(x_i)[d(x_i) - d(x_1)].\end{aligned}$$

From Lemma A.3, $K \geq 0$, we get $R(1) \geq 0$ holds.

For $\boldsymbol{v} = \boldsymbol{v}_5$,

$$\begin{aligned}R(1) &= \sum_{i=1}^{n} v_i b'(x_i)[d(x_i) - d(x_1)] + \sum_{i=1}^{n} v_i b(x_i) d'(x_i)\\ &= b(x_1)d'(x_1) + b(x_2)d'(x_2) - b(x_3)d'(x_3) - b(x_4)d'(x_4)\\ &\quad + b'(x_2)[d(x_2) - d(x_1)] - b'(x_3)[d(x_3) - d(x_1)] - b'(x_4)[d(x_4) - d(x_1)]\\ &\geq b(x_1)d'(x_1) + b'(x_2)[d(x_2) - d(x_1)] - b(x_3)d'(x_3) - b'(x_3)[d(x_3) - d(x_1)] \geq 0.\end{aligned}$$

The last inequality follows from $I \geq 0$ in Lemma A.3.

For $\boldsymbol{v} = \boldsymbol{v}_6$,

$$\begin{aligned}R(1) &= \sum_{i=1}^{n} v_i b'(x_i)[d(x_i) - d(x_1)] + \sum_{i=1}^{n} v_i b(x_i) d'(x_i)\\ &= b(x_1)d'(x_1) + b(x_2)d'(x_2) + b(x_3)d'(x_3) - b(x_4)d'(x_4) - b(x_5)d'(x_5)\\ &+ b'(x_2)[d(x_2) - d(x_1)] + b'(x_3)[d(x_3) - d(x_1)]\\ &- b'(x_4)[d(x_4) - d(x_1)] - b'(x_5)[d(x_5) - d(x_1)]\\ &\geq b(x_1)d'(x_1) + b(x_2)d'(x_2) - b(x_4)d'(x_4)\\ &\quad + b'(x_2)[d(x_2) - d(x_1)] + b'(x_3)[d(x_3) - d(x_1)] - b'(x_4)[d(x_4) - d(x_1)].\end{aligned}$$

From Lemma A.3, $J \geq 0$, we get $R(1) \geq 0$ in this case.

In summary, we have showed that the function Φ is increasing in the direction of these l-vectors. The theorem is thus proved.

3. Discussion

In this paper, we propose a new concept of l-order for parallel systems with n exponential components and conjecture that l-order implies the likelihood ratio order. We have proved the conjecture for $n \leq 5$. As can be seen from the proof, our result can be generalized to parallel systems consisting of other components, such as Weibull components. The conjecture for general n components remains open, although computer simulations seem to confirm its validity.

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Appendix

Lemma A.1. *Let* $b(x) = x/(e^x - 1), d(x) = xe^x/(e^x - 1)$. *Define*

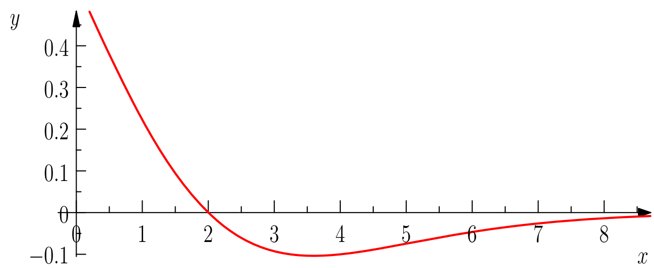
$$G(x, y) = b(y)d'(y) + b'(y)\left[d(y) - d(x)\right].$$

Then, for a fixed $x > 0$, as function of y, $G(x, y)$ has a unique zero, denoted as y_0, and when $0 < y < y_0$, $G(x, y)$ is positive and decreasing while when $y > y_0, G(x,y) < 0$.

Proof. We first prove that $G(x,y) = 0$ has one unique solution. In $G(x, y)$, replace y by x, x by x_1 and rename the resulted function as

$$G(x) = b(x)d'(x) + b'(x)[d(x) - d],$$

where $d = d(x_1)$. The following is the graph of $G(x)$ when $x_1 = 0.2$.



First,

$$G(0^+) = b(0^+)d'(0^+) + b'(0^+)\left[d(0^+) - d\right] = 1 \times \frac{1}{2} - \frac{1}{2}[0 - d] = (1 + d)/2 > 0.$$

Write

$$G(x) = -(e^x - 1)^{-3}F(x),$$

where

$$F(x) = e^{2x}\left[x^2 - (2 + d)x + d\right] + e^x\left[x^2 + (2 + d)x - 2d)\right] + d.$$

When x is big enough, $F(x) > 0$ and thus $G(x) < 0$. It is enough to show that $F(x)$ has one unique zero.

Write $F'(x) = e^x H(x)$, where

$$H(x) = e^x\left[a_0x^2 + b_0x + c_0\right] + x^2 + (4 + d)x + 2 - d,$$

with $a_0 = 2, b_0 = -2(1 + d), c_0 = (d - 2)$. Calculating

$$H'(x) = e^x\left[a_0x^2 + (2a_0 + b_0)x + (b_0 + c_0)\right] + 2x + (4 + d),$$
$$H''(x) = e^x\left[a_0x^2 + (4a_0 + b_0)x + (2a_0 + 2b_0 + c_0)\right] + 2,$$

we get $H'(0) = b_0 + c_0 + 4 + d = 0$, and $H''(0) = 2a_0 + 2b_0 + c_0 + 2 = -3d < 0$.

For $k \geq 3$, the kth derivative of $H(x)$ is

$$H^{(k)}(x) = e^x(a_kx^2 + b_kx + c_k),$$

where $a_k = 2$, $b_k = 4k - 2 - 2d, c_k = 2k^2 - (4 + 3d)k + d - 2$.

Let $k_b = (1 + d)/2, 4k_c = 4 + 3d + \sqrt{9d^2 + 16d + 32}$. When $k \geq k_b, b_k \geq 0$. When $k \geq k_c, c_k \geq 0$. Clearly, $k_c > k_b$, so, when $k \geq k_c$, both b_k and c_k are non negative.

Let k_0 be the smallest number of k such that $b_k \geq 0$ and $c_k \geq 0$. Then, $H^{(k_0-1)}(x)$ is increasing. Notice $H^{(k_0-1)}(0) = c_{k_0-1} < 0, H^{(k_0-1)}(x)$ has one zero point. Denote the zero point as t_0, then, for $0 \leq x \leq t_0, H^{(k_0-2)}(x)$ decreases. Since $H^{(k_0-2)}(0) \leq 0$, we know $H^{(k_0-2)}(x) \leq 0$ for $0 < x \leq t_0$. For $x \geq t_0, H^{(k_0-2)}(x)$ increases. Thus, for $x > t_0, H^{(k_0-2)}(x)$ has one zero point. Therefore, we finally arrive at the conclusion that $F(x)$, and hence $G(x)$, has only one zero point over $x > 0$. Since $G'(x)$ also have one zero, we conclude that $G(x)$ has the properties depicted in the Lemma and showed in the picture. □

Lemma A.2. *For given constants* $\alpha, \beta \geq 0$, *define* $P_{\alpha,\beta}(x,y) = b(x)d'(x) + \alpha b(y)d'(y) + \beta b'(y)[d(y) - d(x)]$. *Then, when* $(\alpha,\beta) = (0,1), (1,1), (1,2), (2,2), (2,3), (3,3), (4,4), P_{\alpha,\beta}(x,y) \geq 0$ *for any* $x, y > 0$.

Proof. From the fact that $b(y)d'(y) \geq 0$, we only need to check the cases of $(\alpha,\beta) = (0,1), (1,2), (2,3)$, and $(4, 4)$.

First, we show that $P_{\alpha,\beta}(x,y) \geq 0$ holds for small x. Since $b(x)d'(x)$ is decreasing, $d(x)$ is increasing, and $b'(x) \leq 0$, so when x is in the range of $(0, x_0)$,

$$\begin{aligned} P_{\alpha,\beta}(x,y) &= b(x)d'(x) + \alpha b(y)d'(y) + \beta b'(y)\left[d(y) - d(x)\right] \\ &\geq b(x_0)d'(x_0) + \alpha b(y)d'(y) + \beta b'(y)\left[d(y) - d(0^+)\right] \\ &= b(x_0)d'(x_0) + \alpha b(y)d'(y) + \beta b'(y)\left[d(y) - 1)\right] \\ &\triangleq Q_{\alpha,\beta}(y). \end{aligned}$$

We will show that for $(\alpha,\beta) = (0,1)$, $(1,2)$, we can take $x_0 = 1$; while $(\alpha,\beta) = (2,3), (4,4)$, we can take $x_0 = 0.4$, to guarantee that $P_{\alpha,\beta}(x,y) \geq 0$ holds for $0 < x < x_0$ and $y > 0$.

Since $b(1)d'(1) > 0.38$ and $b(0.4)d'(0.4) > 0.46$, we have,

$$\begin{aligned} Q_1(y) &\triangleq Q_{0,1}(y) = 0.38 + b'(y)\left[d(y) - 1)\right], \\ Q_2(y) &\triangleq Q_{1,2}(y) = 0.38 + b(y)d'(y) + 2b'(y)\left[d(y) - 1)\right], \\ Q_3(y) &\triangleq Q_{2,3}(y) = 0.46 + 2b(y)d'(y) + 3b'(y)\left[d(y) - 1)\right], \\ Q_4(y) &\triangleq Q_{4,4}(y) = 0.46 + 4b(y)d'(y) + 4b'(y)\left[d(y) - 1)\right]. \end{aligned}$$

By Lemma B.2, $Q_i(y) \geq 0$ for $y \geq 0, i = 1,2,3,4$. Hence, $P_{\alpha,\beta}(x,y) \geq 0$ for $0 < x \leq x_0$.

Now, we prove $P_{\alpha,\beta}(x,y) \geq 0$ when $x > x_0$. Noticing that $d(x)$ is increasing and $b'(x) \leq 0$, thus $P_{\alpha,\beta}(x,y) \geq 0$ when $0 \leq y \leq x$, we only need to prove $P_{\alpha,\beta}(x,y) \geq 0$ when $y > x \geq x_0$. Hence, it suffices to prove that $Q_{\alpha,\beta}(x,y) \geq 0$ when $y > x \geq x_0$.

Denote $a = b(x)d'(x), d = d(x)$, and write $Q_{\alpha,\beta}(x,y)$ simply as $Q(y)$. Recalling that $b(x) = x/(e^x - 1), d(x) = xe^x/(e^x - 1)$, and $d'(x) = [e^x(e^x - 1 - x)]/(e^x - 1)^2$, we have

$$\begin{aligned} Q(y) &= a + \alpha \frac{y}{e^y - 1} \frac{e^y(e^y - 1 - y)}{(e^y - 1)^2} - \beta \frac{ye^y - e^y + 1}{(e^y - 1)^2}\left[\frac{ye^y}{e^y - 1} - d(x)\right] \\ &\overset{\text{sgn}}{=} ae^{3y} + e^{2y}\left[-\beta y^2 + (\alpha + \beta + \beta d)y - (\beta d + 3a)\right] \\ &\quad + e^y\left[-\alpha y^2 - (\alpha + \beta + \beta d)y + 2\beta d + 3a\right] + (-\beta d - a) \\ &\triangleq J(y), \end{aligned}$$

and $J'(y) = e^y J_1(y)$, where

$$\begin{aligned} J_1(y) = 3ae^{2y} &+ e^y\left[-2\beta y^2 + 2(\alpha + \beta d)y + \alpha + \beta - \beta d - 6a\right] \\ &- \alpha y^2 - (3\alpha + \beta + \beta d)y - (\alpha + \beta - \beta d - 3a). \end{aligned}$$

Let $y = x + t, t \geq 0$. Denote $J_1(y) = J_1(x+t)$ by $H_1(t)$. Clearly, at $t = 0$, $H_1(0) \geq 0$. We have

$$\begin{aligned} H_1(t) &= 3ae^{2x+2t} + e^{x+t}\left[a_0(x+t)^2 + b_0(x+t) + c_0\right] + d_0(x+t)^2 + e_0(x+t) + f_0, \\ &\overset{\text{sgn}}{=} e^{2t} + e^t(a_0t^2 + b_0t + c_0) + d_0t^2 + e_0t + f_0 \\ &\triangleq H_2(t), \\ H_2'(t) &\overset{\text{sgn}}{=} e^{2t} + e^t(a_1t^2 + b_1t + c_1) + d_1t + e_1 \triangleq H_3(t), \end{aligned}$$

where

$$a_1 = \frac{1}{2}a_0, \quad b_1 = \frac{1}{2}(2a_0 + b_0), \quad c_1 = \frac{1}{2}(b_0 + c_0), \quad d_1 = d_0, \quad e_1 = \frac{1}{2}e_0,$$
$$a_0 = a_0'/(3ae^x), \quad b_0 = (2a_0'x + b_0')/(3ae^x), \quad c_0 = (a_0'x^2 + b_0'x + c_0')/(3ae^x),$$
$$d_0 = d_0'/(3ae^{2x}), \quad e_0 = (2d_0'x + e_0')/(3ae^{2x}), \quad f_0 = (d_0'x^2 + e_0'x + f_0')/(3ae^{2x}),$$

with $a_0' = -2\beta, b_0' = 2(\alpha + \beta d), c_0' = (\alpha + \beta - \beta d - 6a), d_0' = -\alpha, e_0' = -(3\alpha + \beta + \beta d)$, and $f_0' = -\alpha - \beta + \beta d + 3a$. Specifically,

$$\begin{aligned} a_1 &= -\frac{\beta}{3ae^x}, \\ b_1 &= \frac{-4\beta x - 4\beta + 2\alpha + 2\beta d}{6ae^x}, \\ c_1 &= \frac{-2\beta x^2 + (2\alpha - 4\beta + 2\beta d)x + 3\alpha + \beta + \beta d - 6a}{6ae^x}, \\ d_1 &= -\frac{\alpha}{3ae^{2x}}, \\ e_1 &= -\frac{2\alpha x + 3\alpha + \beta + \beta d}{6ae^{2x}}. \end{aligned}$$

We show in Lemma B.3 that $H_2(0) = 1 + c_0 + f_0 \geq 0$. To prove this lemma, we just need to show $H_3(t) \geq 0$ for $(\alpha, \beta) = (0,1), (1,2), (2,3), (4,4)$.

We first investigate the properties of functions a_1, b_1, c_1, d_1 and e_1 appeared in $H_3(t)$. Since $ae^x = d(x)d'(x)$ is increasing, a_1, d_1 are both increasing, so, to show e_1 is increasing, it is enough to show $e(x) = (2\alpha x + 3\alpha + \beta d)/(ae^{2x})$ is decreasing. In fact,

$$\begin{aligned} e'(x) &\overset{\text{sgn}}{=} (2\alpha + \beta d')dd' - (2\alpha x + 3\alpha + \beta d)(d'2 + dd'' + dd') \\ &\leq (2\alpha + \beta d')dd' - (2\alpha x + 3\alpha + \beta d)(d'2 + dd') \\ &\overset{\text{sgn}}{=} (2\alpha + \beta d')d - (2\alpha x + 3\alpha + \beta d)(d' + d) \\ &= -\alpha d - 2\alpha x d' - 2\alpha x d - 3\alpha d' - \beta d^2 \leq 0. \end{aligned}$$

It turns out that function c_1 is decreasing for the cases of $(\alpha, \beta) = (1,2), (2,3), (3,4)$ but not monotone for $(\alpha, \beta) = (0,1)$, which will be dealt with first.

For the case of $(\alpha, \beta) = (0,1)$, we have

$$H_3(t) = \sum_{k=0}^{\infty} \lambda_k t^k,$$

where $\lambda_0 = 1 + c_1 + e_1, \lambda_1 = 2 + b_1 + e_1 + d_1, \lambda_2 = 2 + a_1 + b_1 + \frac{1}{2}e_1$, and for $k \geq 3$,

$$\begin{aligned} \lambda_k &= \frac{2^k}{k!} + a_1 \frac{1}{(k-2)!} + b_1 \frac{1}{(k-1)!} + c_1 \frac{1}{k!} \\ &\overset{\text{sgn}}{=} 2^k + a_1 k(k-1) + b_1 k + c_1. \end{aligned}$$

We show, for $k \geq 0, \lambda_k \geq 0$. The proofs of $\lambda_0 \geq 0, \lambda_1 \geq 0$, and $\lambda_2 \geq 0$ go to Lemma B.4. For $k \geq 3$, to show $\lambda_k \geq 0$ by induction, we just need to show $I_k(x) = 2^k + 2a_1 k + b_1 \geq 0$, which, in turn, just need to show $2^k + 2a_1 \geq 8 + 2a_1 \overset{\text{sgn}}{=} 4 + a_1 \geq 0$. This is easy to confirm. So, the case of $(\alpha, \beta) = (0,1)$ is checked.

The proof that $c_1(x)$ decreases in the cases of $(\alpha, \beta) = (1,2), (2,3)$, and (3, 4) is not so straightforward, we relegate it to Lemma B.6.

Now we study function b_1. Since $d(0^+) = 1, d'(0^+) = 1/2$, and when $x \to \infty, d(x) - x \to 0$. Hence, $b_1(0^+) = (2\alpha - 2\beta)/3$, and $b_1(\infty) = -\beta/3$.

Function $b_1(x) - b_1(\infty)$ behaves like the function $G(x)$ in Lemma A.1. We show in Lemma B.5 that, when $(\alpha, \beta) = (1,2), b_1(x) \geq -1.25$; when $(\alpha, \beta) = (2,3), b_1(x) \geq -1.75$; when $(\alpha, \beta) = (4,4), b_1(x) \geq -2$. Furthermore, $b_1(x)$ decreases in (0, 1) and increases in $(2, \infty)$.

Table 1. Interval and coefficients of $\theta(t)$ for $(\alpha,\beta)=(1,2)$.

x Interval	a_1	b_1	c_1	d_1	e_1
(1, 2.5]	−0.64	−1.25	−0.25	−0.12	−0.60
(2.5, 3.2]	−0.29	−1.10	−0.37	−0.02	−0.10
(3.2, ∞)	−0.23	−1.02	−0.67	−0.01	−0.05

To prove $H_3(t)\geq 0$ for any $x>x_0$, we divide the interval (x_0,∞) into subintervals and on each of them we find a function $\theta(t)$ such that

$$\begin{aligned}J_3(t)&=e^{2t}+e^t\left[a_1(x)t^2+b_1(x)t+c_1(x)\right]+d_1(x)t+e_1(x)\\&\geq e^{2t}+e^t\left[a_1t^2+b_1t+c_1\right]+d_1t+e_1\\&\triangleq\theta(t)\end{aligned}$$

holds on that interval. The proof of this lemma is completed once we show the positiveness of those functions $\theta(t)$.

The coefficients a_1,b_1,c_1,d_1,e_1 of function $\theta(t)$ are presented in Tables 1–3, corresponding to the cases $(\alpha,\beta)=(1,2),(2,3),(3,3)$, respectively. The proofs of the positiveness of these $\theta(t)$ functions go to Lemma B.1. □

Lemma A.3. *For any* $0<x_1\leq x_2\leq x_3\leq x_4$, *denote*

$$\begin{aligned}I&=b(x_1)d'(x_1)+b'(x_2)[d(x_2)-d(x_1)]-b(x_3)d'(x_3)-b'(x_3)[d(x_3)-d(x_1)],\\J&=b(x_1)d'(x_1)+b(x_2)d'(x_2)+b'(x_2)[d(x_2)-d(x_1)]+b'(x_3)[d(x_3)-d(x_1)]\\&\quad-b(x_4)d'(x_4)-b'(x_4)[d(x_4)-d(x_1)],\\K&=b(x_1)d'(x_1)+b(x_2)d'(x_2)+b(x_3)d'(x_3)\\&\quad+b'(x_2)[d(x_2)-d(x_1)]+b'(x_3)[d(x_3)-d(x_1)]+b'(x_4)[d(x_4)-d(x_1)].\end{aligned}$$

Then, $I\geq 0, J\geq 0, K\geq 0$.

Proof. Consider function

$$G(x)=b(x)d'(x)+b'(x)[d(x)-d(x_1)],\qquad x\geq x_1>0.$$

Clearly, $G(x_1)>0$. By Lemma A.1, for large x, $G(x)\leq 0$, and $G(x)$ has a unique zero. Thus, for any point $x_2\geq x_1$, if $G(x_2)\leq 0$, then $G(x)\leq 0$ for $x\geq x_2$; if $G(x_2)>0$, then $G(x)\leq G(x_2)$ for $x\geq x_2$.

We have

$$I=b(x_1)d'(x_1)-b(x_2)d'(x_2)+G(x_2)-G(x_3).$$

If $G(x_2)>0$, then $G(x_2)-G(x_3)\geq 0$, and clearly $I\geq 0$. Otherwise, if $G(x_2)\leq 0$, then $G(x_3)\leq 0$, and thus

$$\begin{aligned}I&\geq b(x_1)d'(x_1)-b(x_2)d'(x_2)+G(x_2)\\&=b(x_1)d'(x_1)+b'(x_2)[d(x_2)-d(x_1)]\geq 0,\end{aligned}$$

where the last inequality is by Lemma A.2.

For J, in case of $G(x_4)>0$, since $G(x)$ is positive and decreasing in (x_1, x_4), we have

$$\begin{aligned}J&=G(x_1)+G(x_2)+G(x_3)-G(x_4)-b(x_3)d'(x_3)\\&=b(x_1)d'(x_1)-b(x_3)d'(x_3)+G(x_2)+G(x_3)-G(x_4)\geq 0.\end{aligned}$$

In the case of $G(x_4)\leq 0$, denote $m_2=b'(x_2)[d(x_2)-d(x_1)], m_3=b'(x_3)[d(x_3)-d(x_1)]$, and $m=\min\{m_2,m_3\}$. If $m=m_2$, then,

$$\begin{aligned}J&\geq b(x_1)d'(x_1)+b(x_2)d'(x_2)+b'(x_2)[d(x_2)-d(x_1)]+b'(x_3)[d(x_3)-d(x_1)]\\&\geq b(x_1)d'(x_1)+b(x_2)d'(x_2)+2b'(x_2)[d(x_2)-d(x_1)].\end{aligned}$$

Table 2. Interval and coefficients of $\theta(t)$ for $(\alpha, \beta) = (2, 3)$.

x Interval	a_1	b_1	c_1	d_1	e_1
(0.4, 0.5]	−1.46	−1.44	1.89	−0.66	−2.32
(0.5, 0.6]	−1.36	−1.52	1.71	−0.55	−2.03
(0.6, 0.8]	−1.26	−1.52	1.38	−0.46	−1.77
(0.8, 1.0]	−1.10	−1.68	1.10	−0.33	−1.36
(1.0, 1.4]	−0.96	−1.75	0.65	−0.24	−1.05
(1.4, 2.1]	−0.75	−1.75	0.14	−0.13	−0.63
(2.1, 3.1]	−0.52	−1.63	−0.24	−0.05	−0.27
(3.1, 4.0]	−0.35	−1.48	−0.41	−0.02	−0.09
(4.0, ∞)	−0.27	−1.37	−0.84	−0.01	−0.03

Table 3. Interval and coefficients of $\theta(t)$ for $(\alpha, \beta) = (4, 4)$.

x Interval	a_1	b_1	c_1	d_1	e_1
(0.40, 0.46]	−1.95	−1.25	4.05	−1.31	−3.92
(0.46, 0.55]	−1.86	−1.39	3.79	−1.18	−3.62
(0.55, 0.68]	−1.74	−1.55	3.42	−1.01	−3.21
(0.68, 0.82]	−1.59	−1.69	3.06	−0.81	−2.70
(0.82, 0.90]	−1.44	−2.00	2.87	−0.64	−2.25
(0.90, 1.05]	−1.37	−2.00	2.55	−0.56	−2.03
(1.05, 1.35]	−1.24	−2.00	1.99	−0.44	−1.67
(1.35, 1.90]	−1.03	−2.00	1.26	−0.27	−1.14
(1.90, 3.00]	−0.77	−2.00	0.45	−0.12	−0.58
(3.00, 4.50]	−0.48	−1.83	−0.02	−0.03	−0.16
(4.50, 7.00]	−0.31	−1.67	−0.28	−0.01	−0.03
(7.00, ∞)	−0.20	−1.53	−0.67	−0.01	−0.01

If $m = m_3$, then,

$$\begin{aligned} J &\geq b(x_1)d'(x_1) + b(x_2)d'(x_2) + b'(x_2)[d(x_2) - d(x_1)] + b'(x_3)[d(x_3) - d(x_1)] \\ &\geq b(x_1)d'(x_1) + b(x_2)d'(x_2) + 2b'(x_3)[d(x_3) - d(x_1)] \\ &\geq b(x_1)d'(x_1) + b(x_3)d'(x_3) + 2b'(x_3)[d(x_3) - d(x_1)]. \end{aligned}$$

By Lemma A.2, we know, $J \geq 0$ holds.

For K, denote $m_4 = b'(x_4)[d(x_4) - d(x_1)]$. Still, we denote $m = \min\{m_2, m_3, m_4\}$.

If $m = m_4$, then,

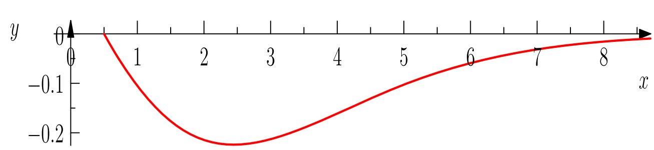
$$\begin{aligned} K &= b(x_1)d'(x_1) + b(x_2)d'(x_2) + b(x_3)d'(x_3) \\ &\quad + b'(x_2)[d(x_2) - d(x_1)] + b'(x_3)[d(x_3) - d(x_1)] + b'(x_4)[d(x_4) - d(x_1)] \\ &\geq b(x_1)d'(x_1) + 2b(x_4)d'(x_4) + 3b'(x_4)[d(x_4) - d(x_1)]. \end{aligned}$$

If $m = m_3$, then,

$$K \geq b(x_1)d'(x_1) + 2b(x_3)d'(x_3) + 3b'(x_3)[d(x_3) - d(x_1)].$$

By Lemma A.2, $K \geq 0$ holds in these cases.

If $m = m_2$, consider function $T(x) = b'(x)[d(x) - d(x_1)]$. Similar to Lemma A.1, we can show that function $T'(x)$ has a unique zero, and hence starting at x_1, $T(x)$ goes down monotonically from zero to negative and goes up monotonically after reaching its minimal value, approaching the x-axis as its asymptotes. Below is the graph of $T(x)$ when $x_1 = 0.5$.



In this case, we have $T(x_4) > T(x_3) > T(x_2)$, and hence,

$$\begin{aligned} K &\geq b(x_1)d'(x_1) + b(x_2)d'(x_2) + b(x_3)d'(x_3) \\ &\quad + b'(x_2)[d(x_2) - d(x_1)] + 2b'(x_3)[d(x_3) - d(x_1)] \\ &= \frac{1}{3}\{b(x_1)d'(x_1) + 2b(x_2)d'(x_2) + 3b'(x_2)[d(x_2) - d(x_1)]\} \\ &\quad + \frac{1}{3}\{b(x_1)d'(x_1) + 2b(x_3)d'(x_3) + 3b'(x_3)[d(x_3) - d(x_1)]\} \\ &\quad + \frac{1}{3}\{b(x_1)d'(x_1) + b(x_2)d'(x_2) + b(x_3)d'(x_3) + 3b'(x_3)[d(x_3) - d(x_1)]\} \\ &\triangleq K_1 + K_2 + K_3. \end{aligned}$$

By Lemma A.2, $K_1 \geq 0, K_2 \geq 0$, and noticing that $b(x_2)d'(x_2) + b(x_3)d'(x_3) \geq 2b(x_3)d'(x_3)$, we conclude that $K_3 \geq 0$ again by Lemma A.2, and hence $K \geq 0$. □

Lemma B.1. *For any 5-tuple $(a_1, b_1, c_1, d_1, e_1)$ listed in Tables 1–3 in the proof of Lemma A.2,*

$$\theta(t) = e^{2t} + e^t\left[a_1t^2 + b_1t + c_1\right] + d_1t + e_1 \geq 0$$

for all $t \geq 0$.

Proof. Since the proofs for other cases are similar, we prove the following two cases as demonstration

$$\begin{aligned} \theta_1(t) &= e^{2t} + e^t[-0.64t^2 - 1.25t - 0.25] - 0.12t - 0.60 \geq 0, \\ \theta_2(t) &= e^{2t} + e^t[-1.44t^2 - 2.00t + 2.87] - 0.64t - 2.25 \geq 0, \end{aligned}$$

which correspond to Table 1, row 1 and Table 3, row 5, respectively.

Write

$$\begin{aligned} \theta_1(t) &= e^{2t} + e^t[-0.64t^2 - 1.25t - 0.25] - 0.12t - 0.60 \\ &\geq e^t(0.5t^2 + t + 1) + e^t[-0.64t^2 - 1.25t - 0.25] - 0.12t - 0.60 \\ &\geq e^t[-0.14t^2 - 0.25t + 0.75] - 0.12t - 0.60. \end{aligned}$$

When $0 < t < 1, -0.14t^2 - 0.25t + 0.75 > 0$, and thus

$$\begin{aligned} \theta_1(t) &\geq [-0.14t^2 - 0.25t + 0.75] - 0.12t - 0.60 \\ &= -0.14t^2 - 0.37t + 0.15. \end{aligned}$$

When $0 < t < 0.3, -0.14t^2 - 0.37t + 0.15 > 0$, and thus $\theta_1(t) \geq 0$. When $t \geq 0.3$, replace t by $t + 0.3$,

$$\begin{aligned} \theta_1(t + 0.3) &= e^{2t+0.6} + e^{t+0.3}\left[-0.64(t + 0.3)^2 - 1.25(t + 0.3) - 0.25\right] - 0.12(t + 0.3) - 0.60 \\ &\stackrel{\text{sgn}}{=} e^{2t} + e^t\left[-\frac{0.64}{e^{0.3}} - \frac{1.634}{e^{0.3}} - \frac{0.6826}{e^{0.3}}\right] - \frac{0.12}{e^{0.6}} - \frac{0.636}{e^{0.6}} \\ &\geq e^{2t} + e^t[-0.475t^2 - 1.22t - 0.51] - 0.066t - 0.35 \\ &\triangleq \vartheta_1(t). \end{aligned}$$

For $\vartheta_1(t)$,

$$\begin{aligned} \vartheta_1(t) &\geq e^t[0.025t^2 - 0.22t + 0.49] - 0.066t - 0.35 \\ &\geq 0.025t^2 - 0.286t + 0.15. \end{aligned}$$

When $0 < t < 0.5, 0.025t^2 - 0.286t + 0.15 > 0$, and hence $\vartheta_1(t) \geq 0$. For $t \geq 0.5$, replace t by $t + 0.5$,

$$\begin{aligned}\vartheta_1(t+0.5) &= e^{2t+1} + e^{t+0.5}\left[-0.475(t+0.5)^2 - 1.22(t+0.5) - 0.51\right] - 0.066(t+0.5) - 0.35\\ &= e^{2t+1} + e^{t+0.5}[-0.475t^2 - 1.695t - 1.23875] - 0.066t - 0.383\\ &\geq e^{2t} + e^t[-0.29t^2 - 1.03t - 0.76] - 0.025t - 0.141\\ &\geq e^t[0.21t^2 - 0.03t + 0.24] - 0.025t - 0.141\\ &\geq 0.21t^2 - 0.055t + 0.099 > 0,\end{aligned}$$

which completes the proof of $\theta_1(t) \geq 0$.

For $\theta_2(t)$,

$$\theta_2(t) \geq e^t[-0.94t^2 - t + 3.87] - 0.64t - 2.25.$$

Since when $0 < t < 1, -0.94t^2 - t + 3.87 > 0,$ and

$$\theta_2(t) \geq -0.94t^2 - 1.64t + 1.62 > 0$$

holds for $0 < t < 0.5$. Thus, $\theta_2(t) \geq 0$ holds for $0 < t < 0.5$. For $t \geq 0.5,$ replace t by $t + 0.5,$

$$\begin{aligned}\theta_2(t+0.5) &\overset{\text{sgn}}{\geq} e^{2t} + e^t[-0.88t^2 - 1.48t + 0.91] - 0.24t - 0.95\\ &\triangleq \vartheta_2(t),\end{aligned}$$

where $A \geq^{\text{sgn}} B$ means $A \geq cB$ for some $c > 0$.

For $\vartheta_2(t),$ when $0 < t < 0.8,$

$$\begin{aligned}\vartheta_2(t) &\geq e^t[-0.38t^2 - 0.48t + 1.91] - 0.24t - 0.95\\ &\geq -0.38t^2 - 0.72t + 0.96.\end{aligned}$$

So, for $0 < t < 0.8, \vartheta_2(t) \geq 0$. For $t \geq 0.8,$ replace t by $t + 0.8,$

$$\begin{aligned}\vartheta_2(t+0.8) &= e^{2t+1.6} + e^{t+0.8}\left[-0.88(t+0.8)^2 - 1.48(t+0.8) + 0.91\right] - 0.24(t+0.8) - 0.95\\ &\overset{\text{sgn}}{\geq} e^{2t} + e^t[-0.40t^2 - 1.30t - 0.38] - 0.05t - 0.24\\ &\geq e^t[0.10t^2 - 0.30t + 0.62] - 0.05t - 0.24\\ &\geq 0.10t^2 - 0.35t + 0.38 \geq 0.\end{aligned}$$

Lemma B.2. *Define*

$$\begin{aligned}Q_1(y) &= 0.38 + b'(y)\left[d(y) - 1)\right],\\ Q_2(y) &= 0.38 + b(y)d'(y) + 2b'(y)\left[d(y) - 1)\right],\\ Q_3(y) &= 0.46 + 2b(y)d'(y) + 3b'(y)\left[d(y) - 1)\right],\\ Q_4(y) &= 0.46 + 4b(y)d'(y) + 4b'(y)\left[d(y) - 1)\right].\end{aligned}$$

Then, $Q_i(y) \geq 0$ *for* $y \geq 0, i = 1, 2, 3, 4.$

Proof. All $Q_i(y)$'s are special case of $Q(y) = a + \alpha b(y)d'(y) + \beta b'(y)[d(y) - 1)]$. Calculate

$$\begin{aligned}Q(y) &= a + \alpha b(y)d'(y) + \beta b'(y)\left[d(y) - 1)\right]\\ &= a + \alpha\frac{y}{e^y - 1}\frac{e^y\left[e^y - (1+y)\right]}{(e^y - 1)^2} - \beta\frac{(y-1)e^y + 1}{(e^y - 1)^2}\left[\frac{ye^y}{e^y - 1} - 1)\right]\\ &\overset{\text{sgn}}{=} ae^{3y} + e^{2y}\left[-\beta y^2 + (\alpha + 2\beta)y - 3a - \beta\right]\\ &\quad + e^y\left[-\alpha y^2 - (\alpha + 2\beta)y + 3a + 2\beta\right] - (a + \beta)\\ &\triangleq \tilde{Q}(y).\end{aligned}$$

$$\begin{aligned}\tilde{Q}'(y) &\overset{\text{sgn}}{=} 3ae^{2y} + e^y\left[-\beta y^2 + 2(\alpha+\beta)y - 6a + \alpha\right]\\ &\quad + \left[-\alpha y^2 - (3\alpha+2\beta)y + 3a - \alpha\right]\\ &\triangleq \hat{Q}(y).\\ \hat{Q}'(y) &= 6ae^{2y} + e^y\left[-\beta y^2 + 2\alpha y - 6a + 3\alpha + 2\beta\right]\\ &\quad - 2\alpha y - (3\alpha + 2\beta)\\ \hat{Q}''(y) &= 12ae^{2y} + e^y\left[-\beta y^2 + 2(\alpha-\beta)y - 6a + 5\alpha + 2\beta\right] - 2\alpha.\end{aligned}$$

We have, $\tilde{Q}(0) = 0, \hat{Q}(0) = 0, \hat{Q}'(0) = 0, \hat{Q}''(0) = 6a + 3\alpha + 2\beta > 0$. So, we just need to show $\hat{Q}''(y) \geq 0$ for $y \geq 0$. The $\hat{Q}_i''(y)$ corresponds to the $Q_i(y)$, $i = 1, 2, 3, 4,$ are

$$\begin{aligned}\hat{Q}_1''(t) &= 4.56e^{2t} + e^t[-t^2 - 2t - 0.28],\\ \hat{Q}_2''(t) &= 4.56e^{2t} + e^t[-2t^2 - 2t + 6.72] - 2,\\ \hat{Q}_3''(t) &= 5.52e^{2t} + e^t[-3t^2 - 2t + 13.24] - 4,\\ \hat{Q}_4''(t) &= 5.52e^{2t} + e^t[-4t^2 + 25.24] - 8.\end{aligned}$$

The proof that $\hat{Q}_i''(t) \geq 0$ is similar to that of the Lemma B.1. □

Lemma B.3. *Let the functions c_0 and f_0 as defined in the proof of Lemma A.3. Then, $H_2(0) = 1 + c_0 + f_0 \geq 0$.*

Proof. We have

$$\begin{aligned}H_2(0) &= 1 + c_0 + f_0\\ &= 1 + (a_0'x^2 + b_0'x + c_0')/(3ae^x) + (d_0'x^2 + e_0'x + f_0')/(3ae^{2x})\\ &\overset{\text{sgn}}{=} 3ae^{2x} + e^x(a_0'x^2 + b_0'x + c_0') + d_0'x^2 + e_0'x + f_0'\\ &= 3ae^{2x} + e^x\left[-2\beta x^2 + 2(\alpha+\beta d)x + (\alpha + \beta - \beta d - 6a)\right]\\ &\quad - \alpha x^2 - (3\alpha + \beta + \beta d)x - \alpha - \beta + \beta d + 3a\\ &= 3\frac{x}{e^x - 1}\cdot\frac{e^x[e^x - (1+x)]}{(e^x-1)^2}e^{2x} + e^x\Bigg\{-2\beta x^2 + 2\alpha x + 2\beta x\frac{xe^x}{e^x-1} + \alpha + \beta\\ &\quad -\beta\frac{xe^x}{e^x-1} - 6\frac{x}{e^x-1}\cdot\frac{e^x[e^x-(1+x)]}{(e^x-1)^2}\Bigg\}\\ &\quad - \alpha x^2 - (3\alpha+\beta)x + \beta x\frac{xe^x}{e^x-1}\\ &\quad - \alpha - \beta + \beta\frac{xe^x}{e^x-1} + 3\frac{x}{e^x-1}\cdot\frac{e^x[e^x-(1+x)]}{(e^x-1)^2}\\ &\overset{\text{sgn}}{=} 3xe^{3x}[e^x - (1+x)] + e^x\left[-2\beta x^2 + 2\alpha x + \alpha + \beta\right](e^x-1)^3\\ &\quad + 2\beta x^2e^{2x}(e^x-1)^2 - \beta xe^{2x}(e^x-1)^2 - 6xe^x[e^x - (1+x)]\\ &\quad + \left[-\alpha x^2 - (3\alpha+\beta)x - \alpha - \beta\right](e^x-1)^3\\ &\quad - \beta x^2e^x(e^x-1)^2 + \beta xe^x(e^x-1)^2 + 3xe^x[e^x-(1+x)]\\ &= e^{4x}[(2\alpha - \beta + 3)x + \alpha + \beta]\\ &\quad + e^{3x}\left[(-3 - \alpha + \beta)x^2 + (-9\alpha + 2\beta - 3)x - 4(\alpha+\beta)\right]\\ &\quad + e^{2x}\left[(3\alpha - 2\beta)x^2 + (15\alpha + 9)x + 6(\alpha+\beta)\right]\end{aligned}$$

$$
\begin{aligned}
&+e^x\left[(-3\alpha+\beta+3)x^2+(-11\alpha-2\beta+3)x-4(\alpha+\beta)\right]\\
&+\alpha x^2+(3\alpha+\beta)x+\alpha+\beta\\
&\triangleq p(x).
\end{aligned}
$$

The proofs for $p(x)\geq 0$ for the four cases are quite the same. So, let us just check the case of $(\alpha,\beta)=(4,4)$. In this case,

$$
\begin{aligned}
p(x)=&\ e^{4x}(7x+8)+e^{3x}(-3x^2-31x-32)\\
&+e^{2x}(4x^2+69x+48)+e^x(-5x^2-49x-32)+4x^2+16x+8.
\end{aligned}
$$

We have,

$$
\begin{aligned}
p'(x)&=e^{4x}(28x+39)+e^{3x}(-9x^2-99x-127)\\
&\quad+e^{2x}(8x^2+146x+165)+e^x(-5x^2-59x-81)+8x+16\\
&=xq(x)+r(x),
\end{aligned}
$$

where,

$$
\begin{aligned}
q(x)&=e^{4x}(28)+e^{3x}(-9x-99)+e^{2x}(8x+146)+e^x(-5x-59)+28,\\
r(x)&=e^{4x}(39)+e^{3x}(-127)+e^{2x}(165)+e^x(-81)+16.
\end{aligned}
$$

It is easy to prove $r(x)\geq 0$ and $q(x)\geq 0$. In fact,

$$
\begin{aligned}
r'(x)&\overset{\text{sgn}}{\geq} e^{3x}-2.443e^{2x}+2.11e^x-0.52,\\
r''(x)&\overset{\text{sgn}}{\geq} e^{2x}-11.63e^x+0.70,\\
r'''(x)&\overset{\text{sgn}}{\geq} e^x-0.82>0.\\
q'(x)&\overset{\text{sgn}}{=}112e^{3x}+e^{2x}(-27x-306)+e^x(16x+300)-5x-64,\\
q''(x)&\overset{\text{sgn}}{=}336e^{3x}+e^{2x}(-54x-639)+e^x(16x+316)-5,\\
q''\prime(x)&\overset{\text{sgn}}{=}1008e^{2x}+e^x(-108x-1332)+16x+332\\
&\overset{\text{sgn}}{=}e^{2x}+e^x[(-108/1008)x-(1332/1008)]+(16/1008)x+(332/1008)\\
&\geq e^{2x}+e^x(-0.11x-1.322)+0.015x+0.329,\\
q^{(4)}&\overset{\text{sgn}}{\geq} e^{2x}+e^x(-0.055x-0.716)+0.007>0.
\end{aligned}
$$

Since at $x=0$, all the functions are non negative, we conclude that $r(x)\geq 0$ and $q(x)\geq 0$, and hence $p'(x)\geq 0$ and $p(x)\geq 0$, and thus $H_2(0)\geq 0$ as claimed. □

Lemma B.4. *Denote* $P_1(x)=1+c_1+e_1, P_2(x)=2+b_1+e_1+d_1, P_3(x)=2+a_1+b_1+\frac{1}{2}e_1$. *Then, in the case of* $(\alpha,\beta)=(0,1), P_i(x)\geq 0,\ i=1,\ 2,\ 3.$

Proof. We have,

$$
\begin{aligned}
P_1(x)&=1+c_1+e_1\\
&=1+\frac{-2x^2+(-4+2dx)+1+d-6a}{6ae^x}+\frac{-1-d}{6ae^{2x}}\\
&\overset{\text{sgn}}{=}6ae^{2x}+e^x\left[-2x^2+(-4+2dx)+1+d-6a\right]-(1+d)\\
&\geq 6ae^{2x}+e^x[-4x+1+d-6a]-(1+d)\\
&\geq 6ae^{2x}+e^x[-4x+1-6a]-1\\
&\triangleq p_1(x),
\end{aligned}
$$

and,

$$p_1'(x) \overset{\text{sgn}}{=} 6(a' + 2a)e^x + (-4x - 3 - 6a - 6a') \overset{\Delta}{=} q_1(x).$$

For $P_2(x)$, we have,

$$\begin{aligned}P_2(x) &= 2 + b_1 + e_1 + d_1 \\ &= 2 + \frac{-4x - 4 + 2d}{6ae^x} + \frac{-1 - d}{6ae^{2x}} \\ &\overset{\text{sgn}}{=} 12ae^{2x} + e^x[-4x - 4 + 2d] - (1 + d) \\ &\geq 12ae^{2x} + e^x[-4x - 4 + d] - 1 \\ &\overset{\Delta}{=} p_2(x),\end{aligned}$$

and,

$$p_2'(x) \overset{\text{sgn}}{=} 12(a' + 2a)e^x + (-4x - 8 + d + d') \overset{\Delta}{=} q_2(x).$$

For $P_3(x)$, we have,

$$\begin{aligned}P_3(x) &= 2 + a_1 + b_1 + \frac{1}{2}e_1 \\ &= 2 - \frac{1}{3ae^x} + \frac{-4x - 4 + 2d}{6ae^x} - \frac{1 + d}{12ae^x} \\ &\overset{\text{sgn}}{=} 24ae^{2x} + e^x[-8x - 12 + 4d] - (1 + d) \\ &\geq 24ae^{2x} + e^x[-8x - 12 + 3d] - 1 \overset{\Delta}{=} p_3(x),\end{aligned}$$

and,

$$p_3'(x) \overset{\text{sgn}}{=} 24(a' + 2a)e^x + (-8x - 20 + 3d + 3d') \overset{\Delta}{=} q_3(x).$$

It is easy to confirm that $p_i(0^+) \geq 0$. So, it is enough to prove that $q_i(x) \geq 0$ for $i = 1, 2, 3$.
Recall that $a = e^x(e^x - 1)^{-4}I, a' = e^x(e^x - 1)^{-4}J$, where,

$$\begin{aligned}I &= x(e^x - 1)[e^x - (1 + x)] = xe^{2x} - x(2 + x)e^x + x(1 + x) \\ J &= (1 - x)e^{2x} + (2x^2 - 2x - 2)e^x + (x^2 + 3x + 1).\end{aligned}$$

Hence,

$$\begin{aligned}q_1(x) &= 6(a' + 2a)e^x + (-4x - 3 - 6a - 6a') \\ &= 6e^{2x}(e^x - 1)^{-4}(J + 2I) + (-4x - 3) - 6e^x(e^x - 1)^{-4}I - 6e^x(e^x - 1)^{-4}J \\ &\overset{\text{sgn}}{=} 6e^{2x}(J + 2I) + (-4x - 3)(e^x - 1)^4 - 6e^xI - 6e^xJ \\ &= e^{4x}(2x + 3) + e^{3x}(-20x - 6) + e^{2x}(6x^2 + 24x) \\ &\quad + e^x(-12x^2 - 8x + 6) + (-4x - 3), \\ q_2(x) &= 12(a' + 2a)e^x + (-4x - 8 + d + d') \\ &= 12e^{2x}(e^x - 1)^{-4}(J + 2I) + (-4x - 8) + xe^x(e^x - 1)^{-1} + e^x[e^x - (1 + x)](e^x - 1)^{-2} \\ &\overset{\text{sgn}}{=} 12e^{2x}(J + 2I) + (-4x - 8)(e^x - 1)^4 + xe^x(e^x - 1)^3 + e^x[e^x - (1 + x)](e^x - 1)^2 \\ &= e^{4x}(9x + 5) + e^{3x}(-60x + 5) + e^{2x}(36x^2 + 41x - 33) \\ &\quad + e^x(12x + 31) + (-4x - 8),\end{aligned}$$

and similarly,

$$q_3(x) \overset{\text{sgn}}{=} e^{4x}(19x+7) + e^{3x}(-124x+23) + e^{2x}(72x^2+87x-87) + e^x(26x+77) + (-8x-20).$$

The proofs of the positiveness of these functions are similar and we check $q_3(x) \geq 0$ as an example. We have

$$\begin{aligned} q_3'(x) &\overset{\text{sgn}}{=} e^{4x}(76x+47) + e^{3x}(-372x-55) + e^{2x}(144x^2+318x-87) \\ &\quad + e^x(26x+103) - 8, \\ q''_3(x) &\overset{\text{sgn}}{=} e^{3x}(304x+264) + e^{2x}(-1116x-537) + e^x(288x^2+924x+144) \\ &\quad + 26x+129, \\ q'''_3(x) &\overset{\text{sgn}}{=} e^{3x}(912x+1096) + e^{2x}(-2232x-2190) + e^x(288x^2+1500x+1068) + 26, \\ q_3^{(4)}(x) &\overset{\text{sgn}}{=} e^{2x}(2736x+4200) + e^x(-4464x-6612) + (288x^2+2076x+2568) \\ &\overset{\text{sgn}}{=} e^{2x}(2.736x+4.200) + e^x(-4.464x-6.612) + (0.288x^2+2.076x+2.568) \\ &> e^{2x}(2.7x+4.2) + e^x(-4.5x-6.7) + 2x + 2.5 \overset{\Delta}{=} r(x). \end{aligned}$$

Since

$$\begin{aligned} r'(x) &= e^{2x}(5.4x+11.1) + e^x(-4.5x-11.2) + 2 \\ r''(x) &= e^{2x}(10.8x+27.6) + e^x(-4.5x-15.7) > 0, \end{aligned}$$

and $r(0)=0, r'(0)=1.9>0,$ we conclude that $r(x) \geq 0,$ and hence $q_3(x) \geq 0.$ □

Lemma B.5. *Let the functions $b_1(x)$ as defined in the proof of Lemma A.2. Then, $b_1(x)$ has the properties as mentioned.*

Proof. We show the case of $(\alpha, \beta) = (4,4)$ as an example. That is, $b_1(x) > -2,$ or, $b_1(x) + 2 > 0$. We have,

$$\begin{aligned} b_1 + 2 &= 2 + \frac{-4\beta x - 4\beta + 2\alpha + 2\beta d}{6ae^x} \\ &\overset{\text{sgn}}{=} 12dd' - 8x - 8 + 8d \\ &\geq 12dd' - 8x \overset{\text{sgn}}{=} 3dd' - 2x \\ &\overset{\text{sgn}}{=} 3e^{2x}(e^x - 1 - x) - 2(e^x-1)^3 \\ &= e^{3x} + 3(1-x)e^{2x} - 6e^x + 2, \end{aligned}$$

which can be easily proved to be ≥ 0. □

Lemma B.6. *The function c_1 defined in the proof of Lemma A.2 decreases over $x>0$ for $(\alpha,\beta) = (1,2),(2,3),(4,4)$.*

Proof. The proofs for the three cases are quite similar, and hence we just prove the case of $(\alpha,\beta) = (4,4)$.

Denote the numerator and denominator of c_1 as N and D, respectively, where $N = -2\beta x^2 + (2\alpha - 4\beta + 2\beta d)x + 3\alpha + \beta + \beta d - 6a$ and $D = dd'$.

Recalling $a = bd', a' = b'd' + bd'', d(x) = xe^x/(e^x-1), d'(x) = e^x[e^x - (1+x)]/(e^x-1)^2, d'' = e^x[e^x(x-2) + (x+2)]/(e^x-1)^3,$ and noticing $d' \geq 0, d'' \geq 0,$ and $x - d \leq 0,$ we have

$$\begin{aligned} c_1' &\overset{\text{sgn}}{=} N'D - ND' \\ &= \left[-4\beta x + (2\alpha - 4\beta) + 2\beta d + 2\beta x d' + \beta d' - 6a'\right] dd' \\ &\quad + \left[2\beta x^2 - (2\alpha - 4\beta)x - 2\beta dx - 3\alpha - \beta - \beta d + 6a\right](d'2 + dd''). \end{aligned}$$

Notice $2\beta x^2 - 2\beta dx \le 0$, we thus have,

$$\begin{aligned} c_1' &\overset{\text{sgn}}{\le} \left[-4\beta x + (2\alpha - 4\beta) + 2\beta d + 2\beta x d' + \beta d' - 6(b'd' + bd'')\right] dd' \\ &\quad + \left[-(2\alpha - 4\beta)x - 3\alpha - \beta - \beta d + 6bd'\right](d'2 + dd'') \\ &= S_1 + S_2, \end{aligned}$$

where

$$\begin{aligned} S_1 &= \left[-4\beta x + (2\alpha - 4\beta) + 2\beta d + 2\beta x d'\right] dd' + (4\beta - 2\alpha)x(d'2 + dd''), \\ S_2 &= -(3\alpha + \beta)(d'2 + dd'') - \beta d^2 d'' + 6d'2(bd' - b'd). \end{aligned}$$

For S_2, we have,

$$\begin{aligned} S_2 &= -(3\alpha + \beta)(d'2 + dd'') - \beta d^2 d'' + 6d'2(bd' - b'd) \\ &\le -(3\alpha + \beta)(d'2 + dd'') + 6d'2(bd' - b'd) \\ &= -16(d'2 + dd'') + 6d'2(bd' - b'd) \\ &\overset{\text{sgn}}{=} -8(d'2 + dd'') + 3d'2(bd' - b'd) \\ &= d'2\left[-8 + 3(bd' - b'd)\right] - 8dd'' \le 0, \end{aligned}$$

which holds due to the fact that $-8 + 3(bd' - b'd) \le 0$.

For S_1, plugging $dd' = xe^{2x}(e^x - 1 - x)(e^x - 1)^{-3}$ and

$$d'2 + dd'' = e^{2x}\left[e^{2x} + e^x(x^2 - 4x - 2) + (2x^2 + 4x + 1)\right]$$

into S_1, we obtain,

$$\begin{aligned} S_1 &\overset{\text{sgn}}{=} \left[-4\beta x + (2\alpha - 4\beta) + 2\beta d + 2\beta x d'\right] x(e^x - 1)(e^x - 1 - x) \\ &\quad + (4\beta - 2\alpha)x\left[e^{2x} + e^x(x^2 - 4x - 2) + (2x^2 + 4x + 1)\right] \\ &\overset{\text{sgn}}{=} [-2x - 1 + d + xd'] x(e^x - 1)(e^x - 1 - x) \\ &\quad + x\left[e^{2x} + e^x(x^2 - 4x - 2) + (2x^2 + 4x + 1)\right] \\ &= \left[-2x - 1 + \frac{xe^x}{e^x - 1} + x\frac{e^x(e^x - 1 - x)}{(e^x - 1)^2}\right] x(e^x - 1)(e^x - 1 - x) \\ &\quad + x\left[e^{2x} + e^x(x^2 - 4x - 2) + (2x^2 + 4x + 1)\right] \\ &\overset{\text{sgn}}{=} \left[(-2x - 1)(e^x - 1)^2 + xe^x(e^x - 1) + xe^x(e^x - 1 - x)\right](e^x - 1 - x) \\ &\quad + (e^x - 1)\left[e^{2x} + e^x(x^2 - 4x - 2) + (2x^2 + 4x + 1)\right] \\ &= \left[-e^{2x} + e^x(-x^2 + 2x + 2) + (-2x - 1)\right][e^x - (1 + x)] \\ &\quad + (e^x - 1)\left[e^{2x} + e^x(x^2 - 4x - 2) + (2x^2 + 4x + 1)\right] \\ &= -xe^{2x} + (x^3 + 2x)e^x - x \\ &\overset{\text{sgn}}{=} -\left[e^{2x} - (x^2 + 2)e^x + 1\right]. \end{aligned}$$

By regular derivative method, it is easy to show $e^{2x} - (x^2 + 2)e^x + 1 \ge 0$. □