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On likelihood ratio ordering of parallel systems with heterogeneous exponential components

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ABSTRACT

This paper provides a partial solution to a long-standing open problem posted by Balakrishnan and Zhao (2013) concerning the likelihood ratio ordering of the lifetimes of parallel systems with two heterogeneous exponential components. The proposed method can be applied to tackle other stochastic ordering problems in a parallel system.

1. Introduction

Stochastic ordering plays an important role in reliability theory, statistical inference, information science, and other applied probability areas. An important example is the study of lifetime of the k-out-of-n systems, where many interesting results have been obtained. See, for instance, Pledger and Proschan (1971), Proschan and Sethuraman (1976), Khaledi and Kochar (2000), Kochar and Xu (2007), Da et al. (2010), Joo and Mi (2010), Torrado and Kochar (2015), Cheng and Wang (2017), among others. For a comprehensive survey of stochastic ordering, see Shaked and Shanthikumar (2007).

Despite for the many progresses made in the study of stochastic ordering in a parallel system, there are long-standing open problems in this field. One of them, documented in Balakrishnan and Zhao (2013), concerns the likelihood order comparison of the lifetime of a parallel systems of two exponential components. Specifically, let $T(\lambda_1, \lambda_2)$ and $T(\mu_1, \mu_2)$ be the lifetimes of two parallel systems with two exponential components whose hazard rates are (λ_1, λ_2) and (μ_1, μ_2), respectively. Here the lifetime distribution of an exponential component with hazard rate λ is $\lambda e^{-\lambda t}$. And let $f_{T(\lambda_1,\lambda_2)}(t)$ and $f_{T(\mu_1,\mu_2)}(t)$ be their probability density functions, respectively. We say that $T(\lambda_1, \lambda_2)$ is larger than $T(\mu_1, \mu_2)$ in the likelihood ratio order, denoted as $T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2)$, if $f_{T(\lambda_1,\lambda_2)}(t)/f_{T(\mu_1,\mu_2)}(t)$ is an increasing function of t.

Assume $\lambda_1 \leq \lambda_2$, $\mu_1 \leq \mu_2$, and $\lambda_1 < \mu_1$. When $\lambda_1 = \lambda_2 = \lambda$, a well-known result is, when $\lambda \leq \mu_1 \leq \mu_2$, $T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2)$. So, in this paper, we assume that $\lambda_1 < \lambda_2$.

Write $(\mu_1, \mu_2) - (\lambda_1, \lambda_2) \overset{\text{sgn}}{=} (1, b)$, where $A \overset{\text{sgn}}{=} B$ if and only if $A = kB$ for a positive k. When $b = -1$, Dykstra et al. (1997) showed that

$$T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2). \tag{1.1}$$

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Zhao and Balakrishnan (2011) showed that (1.1) holds for $-1 \leq b \leq 0$. Yan et al. (2013) proved, under the condition $\lambda_1 \leq \mu_1 \leq \lambda_2$, that b can be extended to $-1 \leq b \leq 1$. Balakrishnan and Zhao (2013) showed by an example that, if $b = 70$, (1.1) does not hold. They went ahead and posted an open problem (Open Problem 1, Balakrishnan and Zhao (2013)) on whether we can extend the range of b for which (1.1) holds.

The objective of this paper is to show that (1.1) holds for $-1 \leq b \leq 10$ but does not hold for $b \geq 12$. The paper is organized as follows. The main result and its proof are presented in Section 2, together with a counter-example showing that the likelihood ratio ordering fails. A short discussion is provided in Section 3.

2. Results and proofs

Theorem 2.1. *Assume $\lambda_1 < \lambda_2$, $\mu_1 \leq \mu_2$, and $\lambda_1 < \mu_1$. Write $(\mu_1, \mu_2) - (\lambda_1, \lambda_2) \overset{\text{sgn}}{=} (1, b)$. Then, for $-1 \leq b \leq 10$, $T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2)$.*

Proof of Theorem 2.1. Let $X = T(\lambda_1, \lambda_2)$ and $Y = T(\mu_1, \mu_2)$. The probability density function of X is $f_X(t) = \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t}$, and that of Y is $f_Y(t) = \mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t} - (\mu_1 + \mu_2)e^{-(\mu_1+\mu_2)t}$. By the definition of likelihood ratio order, $X \geq_{lr} Y$, if and only if, that $f_X(t)/f_Y(t)$ is increasing in $t > 0$, which, in turns, is equivalent to

$$\frac{f_X'(t)}{f_X(t)} - \frac{f_Y'(t)}{f_Y(t)} \geq 0,$$

where,

$$\begin{aligned}\frac{f_X'(t)}{f_X(t)} &= -\frac{\lambda_1^2 e^{-\lambda_1 t} + \lambda_2^2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2)^2 e^{-(\lambda_1+\lambda_2)t}}{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)t}} \\ &= -\frac{\lambda_1^2 e^{\lambda_2 t} + \lambda_2^2 e^{\lambda_1 t} - (\lambda_1+\lambda_2)^2}{\lambda_1 e^{\lambda_2 t} + \lambda_2 e^{\lambda_1 t} - (\lambda_1+\lambda_2)}.\end{aligned}$$

For $0 < x_1 \leq x_2$, define

$$\Phi(x_1, x_2) = \frac{x_1^2 e^{x_2} + x_2^2 e^{x_1} - (x_1+x_2)^2}{x_1 e^{x_2} + x_2 e^{x_1} - (x_1+x_2)} = \frac{x_2 b(x_1) + x_1 b(x_2) - 2}{b(x_1) + b(x_2)},$$

where $b(x) = (e^x - 1)/x$. Then, to prove Theorem 2.1, it suffices to show that $\Phi(x_1, x_2)$ is increasing in the direction of $(1, 10)$. In fact, by Dykstra et al. (1997), $\nabla_{(1,-1)}\Phi > 0$, and thus for any $-1 < b < 10$,

$$\nabla_{(1,b)}\Phi = \frac{10-b}{11}\nabla_{(1,-1)}\Phi + \frac{1+b}{11}\nabla_{(1,10)}\Phi > 0.$$

Since

$$\begin{aligned}\nabla_{(1,10)}\Phi &= \frac{\partial \Phi}{\partial x_1} + 10\frac{\partial \Phi}{\partial x_2} \\ &\overset{\text{sgn}}{=} (x_2 - x_1)b'(x_1)b(x_2) + b(x_1)b(x_2) + b^2(x_2) + 2b'(x_1) \\ &\quad + 10(x_1 - x_2)b'(x_2)b(x_1) + 10b(x_1)b(x_2) + 10b^2(x_1) + 20b'(x_2) \\ &= b(x+t)\{tb'(x) + 11b(x)\} + b^2(x+t) \\ &\quad + b'(x+t)\big\{-10tb(x) + 20\big\} + \big\{10b^2(x) + 2b'(x)\big\}.\end{aligned}$$

Here we reparametrize (x_1, x_2) as $(x, x+t)$ for $t \geq 0$.

Plugging in $b'(x) = (xe^x - e^x + 1)/x^2$,

$$\begin{aligned}\nabla_{(1,10)}\Phi &= \frac{e^x e^t - 1}{x+t}\Big\{tb'(x) + 11b(x)\Big\} + \frac{(e^x e^t - 1)^2}{(x+t)^2} \\ &\quad + \frac{(x+t)e^x e^t - e^x e^t + 1}{(x+t)^2}\Big\{-10tb(x) + 20\Big\} + \Big\{10b^2(x) + 2b'(x)\Big\} \\ &\overset{\text{sgn}}{=} (e^x e^t - 1)^2 + (x+t)(e^x e^t - 1)\Big\{tb'(x) + 11b(x)\Big\} \\ &\quad + \Big\{(x+t)e^x e^t - e^x e^t + 1\Big\}\Big\{-10tb(x) + 20\Big\} \\ &\quad + (x+t)^2\Big\{10b^2(x) + 2b'(x)\Big\} \\ &\overset{\text{sgn}}{=} e^{2t} + e^t\Big\{\alpha_2(x)t^2 + \alpha_1(x)t + \alpha_0(x)\Big\} + \beta_2(x)t^2 + \beta_1(x)t + \beta_0(x) \\ &\overset{\Delta}{=} J_x(t),\end{aligned}$$

where

$$\alpha_2(x) = e^{-x}\Big\{b'(x) - 10b(x)\Big\}, \alpha_1(x) = e^{-x}\Big\{xb'(x) + 21b(x) - 10xb(x) + 20\Big\},$$

Table 1
Coefficients of $\theta_j(t)$.

Interval of x	$(\alpha_{2j},\ \alpha_{1j},\ \alpha_{0j})$	$(\beta_{2j},\ \beta_{1j},\ \beta_{0j})$
$U_1 = (3, 3.4]$	$(-2.94, -2.32, 12.16)$	$(0.81, 5.36, 8.7)$
$U_2 = (3.4, 3.8]$	$(-2.64, -3.19, 11.96)$	$(0.66, 4.96, 9.17)$
$U_3 = (3.8, 4.2]$	$(-2.38, -3.86, 11.76)$	$(0.56, 4.58, 9.45)$
$U_4 = (4.2, 4.7]$	$(-2.16, -4.46, 11.57)$	$(0.46, 4.19, 9.64)$
$U_5 = (4.7, 5.15]$	$(-1.941, -4.97, 11.40)$	$(0.37, 3.82, 9.78)$
$U_6 = (5.15, 5.65]$	$(-1.78, -5.36, 11.28)$	$(0.31, 3.51, 9.86)$
$U_7 = (5.65, 6.25]$	$(-1.62, -5.75, 11.17)$	$(0.25, 3.18, 9.92)$
$U_8 = (6.25, 7.00]$	$(-1.47, -6.12, 11.09)$	$(0.21, 2.85, 9.96)$
$U_9 = (7.00, 8.00]$	$(-1.31, -6.50, 11.04)$	$(0.16, 2.50, 9.98)$
$U_{10} = (8.00, 9.50]$	$(-1.141, -6.90, 11.01)$	$(0.11, 2.10, 9.99)$
$U_{11} = (9.50, 11.95]$	$(-0.96, -7.33, 11.00)$	$(0.07, 1.67, 9.99)$
$U_{12} = (11.95, 16.65]$	$(-0.761, -7.80, 11.00)$	$(0.03, 1.20, 9.99)$
$U_{13} = (16.65, 26.5]$	$(-0.55, -8.25, 11.00)$	$(0.01, 0.75, 9.99)$
$U_{14} = (26.5, 36.35]$	$(-0.35, -8.50, 11.00)$	$(0.00, 0.00, 9.99)$
$U_{15} = (36.35, 46.2]$	$(-0.25, -8.57, 11.00)$	$(0.00, 0.00, 9.99)$
$U_{16} = (46.2, \infty)$	$(-0.20, -9, 11.00)$	$(0.00, 0.00, 9.99)$

$$
\begin{aligned}
&\alpha_0(x) = e^{-x}\left\{11xb(x) + 20x - 22\right\}, \beta_2(x) = e^{-2x}\left\{b'(x) + 10b^2(x)\right\},\\
&\beta_1(x) = e^{-2x}\left\{3xb'(x) - 21b(x) + 20xb^2(x)\right\},\\
&\beta_0(x) = e^{-2x}\left\{10x^2b^2(x) + 2x^2b'(x) - 11xb(x) + 21\right\}.
\end{aligned}
$$

To prove $J_x(t) > 0$, we consider the case $x > 3$ and $0 \le x \le 3$ separately.

For $x > 3$, by Lemma 2.2, $\alpha_2(x)$, $\beta_0(x)$ are increasing, while $\alpha_1(x)$, $\alpha_0(x)$, $\beta_2(x)$, and $\beta_1(x)$ are decreasing. In addition, $\alpha_2(\infty) = 0$, $\alpha_1(\infty) = -9$, $\alpha_0(\infty) = 11$, $\beta_2(\infty) = 0$, $\beta_1(\infty) = 0$, and $\beta_0(\infty) = 10$. We decompose interval $(3, \infty)$ as a disjoint union of 16 subintervals $U_j, j = 1, \ldots, 16$ given in Table 1. When $x \in U_j$, by the monotonicity of $\alpha_i(x)$, $\beta_i(x)$, we can find α_{ij}, β_{ij} such that $\alpha_i(x) \ge \alpha_{ij}$, $\beta_i(x) \ge \beta_{ij}$, $i = 2, 1, 0$, and thus

$$
J_x(t) \ge e^{2t} + e^t\left(\alpha_{2j}t^2 + \alpha_{1j}t + \alpha_{0j}\right) + \beta_{2j}t^2 + \beta_{1j}t + \beta_{0j} \overset{\Delta}{=} \theta_j(t). \tag{2.1}
$$

Table 1 provides the list of α_{ij} and β_{ij}, $i = 2, 1, 0$ associated with interval $U_j, j = 1, \ldots, 16$.

Similar to the case when $x > 3$, we decompose interval $(0, 3\,]$ as a disjoint union of 15 subintervals $V_j, j = 1, \ldots, 15$ given in Table 2. By Lemma 2.2, $\gamma_2(x)$ is increasing, $\gamma_1(x)$ and $\beta_2(x)$ are decreasing. Function $\gamma_0(x)$ increases on $(0, 0.5)$ and decreases on $(0.7, 3)$, and is greater than 10 on $(0.5, 0.7)$. Function $0.5\beta_1(x)$ increases on $(0, 1.5)$ and is greater than 2.8 on $(1.5, 3)$. Thus, when $x \in V_j$, we can find γ_{ij} such that $\gamma_i(x) \ge \gamma_{ij}$, $i = 2, 1, 0$, $\beta_2(x) \ge \delta_{1j}$, and $0.5\beta_1(x) \ge \delta_{0j}$. Thus

$$
H_j(t) \ge e^{2t} + e^t\left(\gamma_{2j}t^2 + \gamma_{1j}t + \gamma_{0j}\right) + \delta_{1j}t + \delta_{0j} \overset{\Delta}{=} \phi_j(t). \tag{2.2}
$$

Table 2 provides the list of γ_{2j}, γ_{1j}, γ_{0j}, δ_{1j}, and δ_{0j}, associated with interval $V_j, j = 1, \ldots, 15$.

We claim that all $\theta_j(t)$ defined in (2.1) and all $\phi_j(t)$ defined in (2.2) are positive when $t > 0$. Due to limitation of space, we prove $\theta_{12}(t) > 0$ as an illustration for the following two reasons. First, all $\theta_j(t)$ and $\phi_j(t)$ have similar graphs, starting at a positive value at $t = 0$, increasing first, and then decreasing, and then increasing again all the way to ∞ (Fig. 1 displays the graph of $\theta_{12}(t)$). Second, since the minimal value of $\theta_{12}(t)$, about 0.016, is the smallest among $\theta_j(t)$ and $\phi_j(t)$, the positivity of $\theta_{12}(t)$ is the most difficult to prove.

By Table 1,

$$
\theta_{12}(t) = e^{2t} + e^t(-0.761t^2 - 7.8t + 11) + 0.03t^2 + 1.2t + 9.99.
$$

When $0 < t \le 1.5$, since $-0.261t^2 - 6.8t + 12 > 0$, we have

$$
\begin{aligned}
\theta_{12}(t) &> e^t(0.5t^2 + t + 1) + e^t(-0.761t^2 - 7.8t + 11) + 0.03t^2 + 1.2t + 9.99\\
&= e^t(-2.61t^2 - 6.8t + 12) > 0.
\end{aligned}
$$

When $t > 1.5$, reparametrizing t as $s + 1.5$ in $\theta_{12}(t)$,

$$
\begin{aligned}
\theta_{12}(t) &= e^{2s+3} + e^{s+1.5}\left[-0.761s^2 - 10.083s - 2.41225\right] + 0.03s^2 + 1.29s + 11.8575\\
&\overset{\text{sgn}}{=} e^{2s} + e^s\left[-0.761e^{-1.5}s^2 - 10.083e^{-1.5}s - 2.41225e^{-1.5}\right]\\
&\quad + 0.03e^{-3}s^2 + 1.29e^{-3}s + 11.8575e^{-3}\\
&> e^{2s} + e^s\left[-0.1699s^2 - 2.2499s - 0.5383\right] + 0.00149s^2 + 0.06422s + 0.5903\\
&\overset{\Delta}{=} \vartheta_1(s).
\end{aligned}
$$

For $0 \le s \le 0.4$, $\vartheta_1(s) > e^s(0.33s^2 - 1.2499s + 0.4617) > 0$. Equivalently, for $1.5 \le t \le 1.9$, $\theta_{12}(t) > 0$.

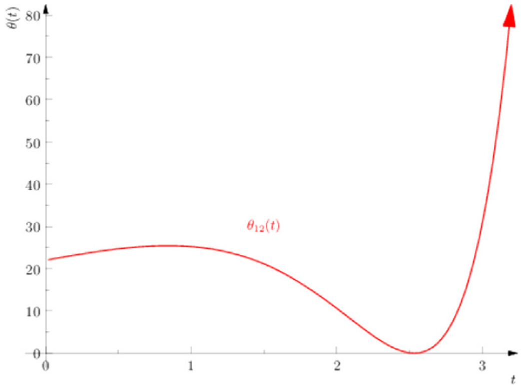

Fig. 1. Plot of $\theta_{12}(t)$.

Table 2
Coefficients of $\phi_j(t)$.

Interval of x	$(\gamma_{2j},\ \gamma_{1j},\ \gamma_{0j})$	$(\delta_{1j},\ \delta_{0j})$
$V_1 = (0, 0.5]$	$(-4.75, 4.93, 9.5)$	$(6.39, -10.5)$
$V_2 = (0.5, 0.7]$	$(-3.73, 3.34, 10)$	$(5.37, -1.73)$
$V_3 = (0.7, 1.2]$	$(-3.40, 0.36, 9.98)$	$(3.49, 0.07)$
$V_4 = (1.2, 1.5]$	$(-2.74, -0.84, 9.19)$	$(2.75, 2.41)$
$V_5 = (1.5, 1.9]$	$(-2.43, -1.97, 8.09)$	$(2.04, 2.80)$
$V_6 = (1.9, 2.2]$	$(-2.10, -2.57, 7.31)$	$(1.66, 2.80)$
$V_7 = (2.2, 2.4]$	$(-1.89, -2.89, 6.82)$	$(1.45, 2.80)$
$V_8 = (2.4, 2.55]$	$(-1.77, -3.08, 6.48)$	$(1.32, 2.80)$
$V_9 = (2.55, 2.65]$	$(-1.69, -3.20, 6.27)$	$(1.24, 2.80)$
$V_{10} = (2.65, 2.75]$	$(-1.64, -3.31, 6.06)$	$(1.17, 2.80)$
$V_{11} = (2.75, 2.80]$	$(-1.59, -3.36, 5.96)$	$(1.13, 2.80)$
$V_{12} = (2.80, 2.85]$	$(-1.56, -3.40, 5.86)$	$(1.10, 2.80)$
$V_{13} = (2.85, 2.90]$	$(-1.53, -3.45, 5.77)$	$(1.07, 2.80)$
$V_{14} = (2.90, 2.95]$	$(-1.52, -3.49, 5.67)$	$(1.04, 2.80)$
$V_{15} = (2.95, 3.00]$	$(-1.491, -3.526, 5.586)$	$(1.014, 2.80)$

For $s > 0.4$, reparametrizing s as $w + 0.4$ in $\vartheta_1(s)$, by the same process, we have

$$\begin{aligned}\vartheta_1(s) &= e^{2w+0.8} + e^{w+0.4}(-0.1699w^2 - 2.38582w - 1.465444)\\ &\quad + 0.00149w^2 + 0.065412w + 0.6162264\\ &\overset{\text{sgn}}{>} e^{2w} + e^{w}(-0.113888w^2 - 1.599263w - 0.982317)\\ &\quad + 0.000669w^2 + 0.029391w + 0.276888\\ &\triangleq \vartheta_2(w),\end{aligned}$$

where $A \overset{\text{sgn}}{>} B$ means $A > cB$ for some $c > 0$.

By the fact that $\vartheta_2^{(6)}(w) > 0$ and $\vartheta_2^{(k)}(0) > 0$ for $k = 3, 4, 5$, we conclude that $\vartheta_2^{(3)}(w) > 0$.

We claim that $\vartheta_2'(w)$ has exactly one zero over interval $(0, \infty)$. Otherwise, assume $\vartheta_2'(w)$ has more than one zero. Noticing that $\vartheta_2'(0) < 0$ and $\vartheta_2'(\infty) > 0$, we conclude that $\vartheta_2'(w)$ has at least three zeros, and thus, $\vartheta_2^{(3)}(w)$ has at least one zero, which contradicts the fact that $\vartheta^{(3)}(w) > 0$, which we just proved.

Since $\vartheta_2'(w)$ has exactly one zero and $\vartheta_2'(0) < 0$, $\vartheta_2'(0.63) = -0.0121917 < 0$, then in $(0, 0.63)$, $\vartheta_2'(w) < 0$, and thus $\vartheta_2(w)$ is decreasing. Therefore, when $0 \le w \le 0.63$, $\vartheta_2(w) > \vartheta_2(0.63) > 0.5 \times 10^{-5} > 0$. Equivalently, when $1.9 \le t \le 2.53$, $\theta_{12}(t) > 0$.

For $w > 0.63$, reparametrizing w by $u + 0.63$,

$$\begin{aligned}\vartheta_2(w) &\overset{\text{sgn}}{>} e^{2u} + e^{u}(-0.060656u^2 - 0.928181u - 1.083855)\\ &\qquad + 0.000189u^2 + 0.008575u + 0.083867 \triangleq \vartheta_3(u).\end{aligned}$$

Taylor expansion yields $\vartheta_3(u) = \sum_{n=0}^{\infty} \alpha_n t^n$, where $\alpha_0 = 0.000012$, $\alpha_1 = -0.003461$, $\alpha_2 = 0.4694245$.

For $n \ge 3$, it is easy to prove by the Mathematical Induction Principle that

$$\alpha_n = \frac{1}{n!}\left[2^n - 0.060656n(n-1) - 0.928181n - 1.083855\right] > 0.$$

Thus, $\vartheta_3(u) \geq \alpha_2 u^2 + \alpha_1 u + \alpha_0$. Since the determinant $\Delta = \alpha_1^2 - 4\alpha_2\alpha_0 = -1.055385 \times 10^{-5} < 0$ and $\alpha_2 > 0$, we conclude that, $\vartheta_3(u) > 0$ for all $u \geq 0$. Equivalently, when $t \geq 2.53$, $\theta_{12}(t) > 0$.

To sum up, we have proved that $\theta_{12}(t) > 0$ for all $t > 0$ as claimed. □

Lemma 2.2. *Let $\alpha_j(x)$, $\beta_j(x)$, $\gamma_j(x)$, $j = 0, 1, 2$, be defined in the proof of Theorem 2.1. Then,*

1. $1 + \alpha_0(x) + \beta_0(x) > 0$ *for* $x > 0$.
2. *When* $x > 3$, $\alpha_2(x)$, $\beta_0(x)$ *increase,* $\alpha_1(x)$, $\alpha_0(x)$, $\beta_2(x)$, *and* $\beta_1(x)$ *decrease.*
3. *When* $0 < x \leq 3$, $\gamma_2(x)$ *increases,* $\gamma_1(x)$ *and* $\beta_2(x)$ *decrease;* $\gamma_0(x)$ *increases on* $(0, 0.5)$, *decreases on* $(0.7, 3)$; *and* $\gamma_1(x) > 10$ *on* $(0.5, 0.7)$; $0.5\beta_1(x)$ *increases on* $(0, 1.5)$, *and is larger than 2.8 on* $(1.5, 3)$.

The proof of Lemma 2.2 is routine and hence is omitted.

Example 2.3. Recall $f_X(t) = \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t}$, $f_Y(t) = \mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t} - (\mu_1 + \mu_2)e^{-(\mu_1+\mu_2)t}$. Denote $D(t) = f_X'(t) f_Y(t) - f_X(t) f_Y'(t)$. Choose $(\lambda_1, \lambda_2) = (1, 2)$, $(\mu_1, \mu_2) = (1.1, 3.2)$, $D(1.2) = -0.047 < 0$, which indicates that $T(1, 2) \not\geq_{lr} T(1.1, 3.2)$. Thus, in the direction $(1, 12)$, there is no likelihood ratio order between $T(\lambda_1, \lambda_2)$ and $T(\mu_1, \mu_2)$.

3. Discussion

Determining the direction $(1, b) \overset{\text{sgn}}{=} (\mu_1, \mu_2) - (\lambda_1, \lambda_2)$ in which $T(\lambda_1, \lambda_2) \geq_{lr} T(\mu_1, \mu_2)$ is an extremely challenging problem with a best previous record of $b = 1$. If likelihood ratio ordering holds for a direction $(1, b)$ with $b > -1$, then it also holds for any direction $(1, b')$ with $-1 < b' < b$. In fact, by assumption we have $\nabla_{(1,b)}\Phi > 0$, and by Dykstra et al. (1997) we have $\nabla_{(1,-1)}\Phi > 0$, and thus $\nabla_{(1,b')}\Phi = \frac{b-b'}{b+1}\nabla_{(1,-1)}\Phi + \frac{1+b'}{b+1}\nabla_{(1,b)}\Phi > 0$. The above observation also implies, by the contradiction argument, that if likelihood ratio ordering does not hold for a direction $(1, c)$ for some $c > -1$, then it does not hold for any direction $(1, c')$ with $c' > c$ neither. Thus, there must exist a unique $b^* > 0$ such that likelihood ratio ordering holds for $-1 \leq b < b^*$ but does not hold for all $b > b^*$ (the case when $b = b^*$ has to be discussed separately since b^* is a critical point). Our results imply that $10 \leq b^* < 12$, while the best result before us was $1 \leq b^* < 70$. This indicates that our paper greatly improves the existing result.

The main difficulty of this type of problems is to prove that a non-monotonic function is positive. We propose a systematic method to handle this challenge. The main idea is to divide the domain into small intervals and use reparametrization to localize the function around zero where e^t can be well approximated by a quadratic polynomial whose positiveness is easy to verify. We did not spend effort to optimizing our decompositions in Tables 1 and 2, which means, with finer decompositions, the conclusion could be further strengthened. Our numerical explorations indicated that the largest b^* is around 10.78. Furthermore, we believe the theoretical result presented in this paper can provide some kinds of guidance to compare the qualities or reliabilities of two parallel systems. It can also be useful in testing hypothesis concerning these systems.

CRediT authorship contribution statement

Jiantian Wang: Conceptualization, Methodology, Writing - original draft. **Bin Cheng:** Methodology, Validation, Writing - original draft, Writing - review & editing.

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