Limb-Leaf designs for adaptive exploration of the dose-response curve

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ABSTRACT

We propose a two-stage strategy, called the Limb-Leaf method, to explore the dose-response curve using dose promotion and addition in the context of adaptive seamless Phase II/III trials. Strong control of the overall type 1 familywise error rate of the proposed method is enforced by the closed testing principle. The design constants are determined to minimize the risk-adjusted expected total sample size while maintaining a target power. In the case of a nonmonotonic dose response curve where more doses are required to adequately explore the curve, substantial savings in sample size are achieved compared with a traditional strategy which offers only selection and promotion from among initial first stage doses.

1. Introduction

The traditional process of drug development generally consists of four phases: Phase 1, to find which doses can be tolerated, particularly the maximum tolerated dose (MTD); Phase 2, to determine the biological activity and adverse event rates of the tolerated doses; Phase 3, to determine efficacy of a selected dose; and Phase 4, after regulatory approval of the drug, as a review of safety and other long-term results. Traditionally, Phase 3 is run and analyzed independently of Phase 2, i.e., the Phase 2 results are not used in the final determination of efficacy.

One method of reducing the large costs in time, money, and patient exposure of this process is to merge phases together and to eliminate the gaps and delays between them. In general, a seamless design combines the objectives of multiple phases of the development process into a single trial. In particular, it is often possible to meet the objectives of Phases 2 and 3 within one less costly, combined study.

An adaptive seamless design is one that: (1) combines the objectives of different stages, (2) allows modification of the trial based on emerging data, and (3) is inferentially seamless in the sense that the final analysis combines data from before and after any adaptation while maintaining control over the type 1 error rate. A landmark two-stage adaptive seamless design was proposed by Thall et al. [19] (henceforth, the TSE Design). There a first stage is used to select the best (or apparently best) of several candidate treatments and the second stage focuses only on the selected treatment. Both stages include a control arm and the data from both stages are pooled for the final comparison between control and selected arm. This design in its original formulation applies only to binary outcomes, however the TSE template is easily modified to other outcome distributions, for instance normal outcomes as described by Jennison and Turnbull [7,8]. An important generalization that includes multiple stages and the use of a test based on the score statistic was proposed by Stallard and Todd [16], and Todd and Stallard [20]. This method accommodates a general endpoint, which, for instance, could be normal, binary, or a time to an event.

Another route for the development of adaptive seamless designs has been through the adaptive P-value combination tests used by Bauer and Köhne [2]. This approach allows information from earlier stages to be combined with that of later stages, and treatment selection at adaptive interim analyses to be based on all previous information from inside and outside the trial. Midtrial modifications are possible without inflating the familywise type 1 error rate. The key ideas are: the construction of P-values with conditionally (sub)uniform distributions given the previous stages of the experiment, the pooling of evidence across stages using prespecified combination rules, and the use of a closed testing procedure to control the familywise error rate for the multiple hypotheses under study. The method of adaptive combination tests is very general and [14] shows how it includes group sequential tests, the two stage TSE design, and further generalizations as special cases.

2. A proposal for adaptive exploration

Experiments can benefit from a more structured exploration strategy when the dose-response curve is not assumed to be monotonic, particularly so in the case where it is still assumed to be unimodal. This
possibility is especially plausible when dealing with a combined efﬁ-
cacy/toxicity or beneﬁt/cost endpoint, or when dealing with a com-
ination therapy such that no single ordering of the doses may be pos-
sible. We present the following example of a real-world setting in which such an exploration strategy might be applied.

2.1. An example based on the QALS trial

The U.S. Food and Drug Administration has approved only one 
drug, Riluzole, for the treatment of the devastating neurodegenerative 
disease Amyotrophic Lateral Sclerosis (ALS). Beneﬁts in patient func-
tion and survival are small and safety and tolerability of Riluzole 
especially in regard to liver toxicity are major considerations. There is 
the hope that the progression of ALS may one day be slowed or stopped 
by new medications or combinations of drugs.

Methods of action of Riluzole or other possible therapies are not 
fully elucidated; they are complex and may involve multiple mechan-
isms as reported, for instance, by Hubert et al. [5], Noh et al. [15], and 
Beal et al. [3]. In such settings the expectation that an investigational 
drug would have a monotonic dose response curve with increasing 
patient benefit up to a well-deﬁned maximum tolerated dose is es-
pecially problematic. Indeed an incorrect assumption of monotonicity 
in the dose response is a favored explanation for the failure of a recent 
major ALS study as reported by Ludolph and Jesse [12].

The QALS trial published by Kaufmann et al. [9] was undertaken to 
investigate high doses of Coenzyme Q10 as a possible therapy for ALS. 
It was not assumed that the dose response would be monotonic and 
toxicity was carefully monitored. The study was designed in two stages, 
a selection stage followed by a futility test, also known as a non-superi-
ority test (see, e.g., [10]). Two doses of CoQ10 (1800 and 2700 mg 
per day) together with a placebo arm began the first stage of the study. 
After an interim analysis the apparently better performing dose was 
selected to continue and additional recruitment to that arm and the 
placebo arm took place in the second stage. The final test statistics 
involved data pooled over both stages under appropriate control for 
selection bias and type 1 error.

The outcome of the first stage of the trial was that the higher dose 
did better than the lower dose on the outcome measure (ALSFRSr, the 
ALS functional rating scale, revised) and had high tolerability. After 
continuation into the second stage the test statistic associated with this 
higher dose was nominally sufﬁcient to avoid a declaration of futility. 
The investigators nevertheless did not consider the evidence promising 
ough to give it full endorsement. Further details are given in 
Kaufman et al. [9].

Notwithstanding the importance of certain key differences of the 
QALS trial, especially its aim to test a futility hypothesis rather than 
as a superiority hypothesis, it is easy to imagine that in this or a similar 
study, a conventional superiority hypothesis might instead be the goal. 
In such a case, for example, it might have been considered worthwhile 
to allow further exploration in the second stage around the apparently 
better ﬁrst stage dose. Speciﬁcally, had there been an option to add 
higher doses beyond 2700 mg per day or to explore both above and 
below this dose level, this freedom and ﬂexibility might have been at-
ttractive to investigators. It is perhaps possible that an eﬃcacious dose 
might have been among those added, and good enough to earn a ful-
en endorsement. We would like to make such further options available 
to investigators by design in especially diﬃcult disease areas like ALS.

2.2. Stagewise adaptive exploration

In exploring the dose-response relationship, particularly a non-
monotonic one, it is important to distinguish between \(d^*\), a dose with 
the maximum possible effect, and \(d^\ast\), its estimated value. The corre-
ponding eﬀects of these doses, \(d^\ast\) and \(d^\ast\), say, could be diﬀerent in a 
meaningful way, with \(d^\ast > d^\ast\). Similarly, if \(d^\ast\) denotes a dose with a 
given desired eﬀect (not necessarily maximum), \(|d^\ast - d^\ast|\) could be 
undesirably large. On the other hand, at least heuristically, the more 
closely \(d^\ast\) approximates \(d^\) the greater the chance that the study will 
reject the global null hypothesis, \(H_0: \theta_i \leq 0\) for all doses \(d\), and more 
importantly, the closer the ﬁnal recommended dose will be to that 
which gives patients the desired (or maximum) beneﬁt. These consid-
erations provide strong motivation to better explore the dose-re-
response curve in adaptive seamless designs.

One possibility for better exploration of a non-monotonic dose-re-
response curve in a single-stage selection strategy like the TSE Design 
would be to include a large number of closely spaced doses in the ﬁrst 
stage. However, there are reasons to expect performance to suﬀer with 
this approach. First, the true \(d^\) may be hard to identify because it will 
have many competitors, some with nonzero eﬀects. Also, a large 
number of ﬁrst stage patients will have to be randomized to areas of the 
dose response curve that are not relevant to the ﬁnal recommendation. 
Second, under the global null hypothesis of no treatment eﬀect at any 
dose, many patients will have been treated before an early stopping 
decision can be made. We note that under either hypothesis, treating an 
excessive number of patients with an ineffective treatment or with in-
eﬀective doses of an otherwise worthwhile treatment is ethically un-
 desirable. Some of these issues are mentioned by Thall et al. [19], but 
they play less of a role in the case where there are only a few possible 
doses to consider with broad spacing between them.

In this paper we introduce a two-stage selection procedure called 
the “Limb-Leaf” design in which second stage doses not only are pro-
moted from a modest number of ﬁrst stage candidates but also may be 
added in response to ﬁrst stage results. We aim to improve the esti-
mation of \(d^\) and to use resources more eﬃciently, this being particu-
larly so under the global null hypothesis, where the probability of early 
stopping should be large. Such promotion and addition decisions can be 
based on all the available information, including eﬃcacy and toxicity, 
whether it comes from within the study or from an outside source.

Although the Limb-Leaf design we present here achieve pre-spe-
ciﬁed performance characteristics in the case of a non-monotonic dose-
response curve, it actually comprises a more general approach to 
structured exploration of response functions.

3. Method

Below we assume without essential loss of generality that the data 
are normal with variance \(\sigma_i^2\) known. Extensions to other data types are 
mentioned with further detail given in the Appendix.

3.1. The TSE method

The TSE design has two stages; the ﬁrst stage assigns subjects to all 
candidate doses plus the control, and the second stage studies only the 
best performing dose from the ﬁrst stage against the control. There is an 
opportunity to stop for futility using a eﬃcacy value after the ﬁrst stage, 
and the ﬁnal decision is made by whether the combined measure of eﬀect 
of the selected treatment exceeds a second eﬃcacy value.

A version of the TSE design using normal outcomes is described as 
follows. Let the test doses in the experiment be denoted as \(d_1, \ldots, d_i\), with 
eﬀects relative to control dose \(d_0\), of \(\theta_1, \ldots, \theta_i\). We assume that at either 
stage, the outcomes at a given dose \(d_i\) are independent and identically 
distributed (i.i.d.) as a normal random variable with mean \(\mu_i\) and var-
iance \(\sigma_i^2\), \(j = 0, 1, \ldots, I\). The design proceeds in two stages:

Stage 1. Randomize \((I + 1)n_1\) patients equally to \(d_0, d_1, \ldots, d_i\). Let 
\(T_i = \max_{1 \leq i \leq t} T_{i,p}\) where for each \(i, \tilde{T}_i = (X_i - X_0) / \sqrt{2\sigma_i^2}\), \(X_0\) is 
the sample mean for the control \(d_0\) and \(X_i\) the sample mean for 
dose \(d_i\), \(i = 1, \ldots, I\) at stage 1. If \(T_1 > y_1\), then continue by selecting 
the treatment \(d_i\) having the greatest observed eﬀect, \(\tilde{T}_i\), into a 
second stage. If \(T_1 \leq y_1\) then stop and accept \(H_0\) of no eﬀect on any 
dose.

Stage 2. Randomize \(2n_2\) additional patients equally to \(d_i\) and \(d_0\). Let
The Limb-Leaf design

3.2.1. Limb-Leaf structure and locatable effect

Let \( S = \{d_1, \ldots, d_i\} \) be a prespecified collection of test doses to be investigated as in a TSE design. A Limb-Leaf structure for \( S \) is a re-arrangement of the doses in \( S \) as

\[
S = \{l_1, b_1, \ldots, l_m, b_m, \ldots, l_k, b_k, \ldots, l_{kk}, b_{kk}\},
\]

where for each “limb” dose \( L_k, k = 1, \ldots, K \), there is an associated neighborhood also including \( m \) leaf doses, \( l_{ki}, \ldots, b_{ki} \). Let \( \Theta_\nu \) denote the dose-response configuration, be the collection of effects associated with Limb-Leaf system \( S \) that is,

\[
\Theta_\nu = \{\hat{\theta}_{l_{ki}}, \hat{\theta}_{b_{ki}}, \ldots, \hat{\theta}_{l_{mk}}, \hat{\theta}_{b_{mk}}, \ldots, \hat{\theta}_{l_{kk}}, \hat{\theta}_{b_{kk}}\}.
\]

Generally, the effects \( \hat{\theta}_{l_{ki}}, \ldots, \hat{\theta}_{b_{mk}} \) on the respective leaves \( l_{ki}, \ldots, b_{ki} \) would be assumed to be somewhat similar but not identical to that of their associated limb dose \( L_k \).

In this paper, we assume that the dose-response configuration under investigation has a “locatable” effect, whose definition is as follows.

**Definition 1. A dose-response configuration \( \theta_\nu \) is defined to have a locatable effect with respect to the Limb-Leaf System \( S = \{l_1, b_1, \ldots, l_m, b_m, \ldots, l_k, b_k, \ldots, l_{kk}, b_{kk}\} \) and the response levels \( \delta = (\delta_1, \delta_2, \delta_3) \), with \( \delta_1 < \delta_2 < \delta_3 \) where \( \delta_i \) is considered to be the smallest desired level of effect, if the following two conditions hold:

1. The effects on each limb \( \hat{\theta}_{l_{ki}} \) for \( k = 1, \ldots, K \), are either less than or equal to \( \delta_1 \), or greater than or equal to \( \delta_2 \), with at least one \( k \) such that \( \delta_{l_{ki}} \geq \delta_1 \).
2. For any limb \( L_k \) with \( \delta_{l_{ki}} \geq \delta_2 \), each effect in its neighborhood is either greater than or equal to \( \delta_2 \) or less than or equal to \( \delta_3 \) with at least one effect greater than or equal to \( \delta_3 \).

Intuitively, a dose-response configuration \( \Theta_\nu \) with a locatable effect means that the shape of the dose response curve should permit exploration in stages, first on a coarse level and second for finer level adjustments, in order to successfully identify a dose of desired effect level \( \delta_2 \) or better.

**Proposition 1. Any given dose-response configuration \( \Theta_\nu \) has a locatable effect with respect to the Limb-Leaf System \( S = \{l_1, b_1, \ldots, l_m, b_m, \ldots, l_k, b_k, \ldots, l_{kk}, b_{kk}\} \) and some vector of response levels \( \delta = (\delta_1, \delta_2, \delta_3) \), with \( \delta_1 < \delta_2 < \delta_3 \).

The proof of this result is given in the Appendix. It follows that for prespecified \( \Theta_\nu \), \( \delta \), and \( \delta \), failure to meet the definition of locatability may be considered as a misspecification of \( \delta \), which is a useful perspective in examining the performance of the Limb-Leaf approach under violations of assumptions. This does not, however, mean that selection decisions for \( S \) and \( \delta \) are unimportant; a selection of \( S \) and \( \delta \) that are not only formally correct but appropriate to the underlying dose-response relationship is key for a design to have good performance characteristics. Guidance on how \( S \) and \( \delta \) may be chosen and sources of information for this decision are offered in Sections 5 and 6.

3.2.2. The plan of a Limb-Leaf design

A Limb-Leaf method proceeds as follows:

**Step 1.** Prespecify a vector \( \mathbf{c} = (c_1, c_2) \), \( c_1 < c_2 \), representing different levels of a test statistic, and a vector of weights \((w_1, w_2)\) such that \( w_1, w_2 \geq 0 \), with \( w_1 + w_2 = 1 \).

**Step 2.** Randomize \( n_{LL} \) patients to each limb and control. Let \( \hat{\theta}_{l_{ki}} = X_{l_{ki}} - X_{b_{ki}} \) denote the first stage estimate and \( Z_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \) denote the first stage test statistic of the effect of dose \( L_k, k = 1, \ldots, K \). The limb with the greatest first stage statistic, denoted by \( L^* \), will have an estimate denoted by \( \hat{\theta}_{l_{ki}} \) and a test statistic denoted by \( Z_{l_{ki}} \).

Note that \( k^* \) is a random variable.

**Step 3.** There are 3 possibilities.

(i) If \( \hat{\theta}_{l_{ki}} \leq c_1 \), then the study stops for futility.

(ii) If \( c_1 < \hat{\theta}_{l_{ki}} \leq c_2 \) the experiment continues to Stage 2 with \( L^* \), its leaves \( l_{k^*}, \ldots, b_{k^*} \), and the control dose \( L_0 \). Randomize \( n_{LL} \) patients to each of \( L^* \) and control \( L_0 \) and repeat the experiment for \( l_{k^*}, \ldots, b_{k^*} \).

Let \( \hat{\theta}_{l_{ki}} = X_{l_{ki}} - X_{b_{ki}} \) and \( \hat{\theta}_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \) and \( Z_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \)

and \( Z_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \) and \( \hat{\theta}_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \) and \( Z_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \) and \( \hat{\theta}_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \) and \( Z_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \) and \( \hat{\theta}_{l_{ki}} = \hat{\theta}_{l_{ki}} / \sqrt{\frac{\sigma^2}{n_{LL}}} \)

If \( \hat{\theta}_{l_{ki}} < c_2 \), then we reject \( H_{0, l_{ki}}: \hat{\theta}_{b_{ki}} \leq 0 \) and estimate \( \delta^* \) as \( L^* \), if and only if

\[
w_1 \Phi^{-1}[F_\nu(Z_{l_{ki}})] + w_2 \Phi^{-1}[G_{\nu_{LL}}(\max(\hat{\theta}_{l_{ki}} - Z_{l_{ki}}, \hat{\theta}_{l_{ki}}))] > \alpha
\]

Here \( \nu_{LL} \) is the kth smallest order statistic of \( \{Z_{l_{ki}}, \ldots, Z_{l_{ki}}\} \) and it is understood that for \( k = 0 \) the maximum in the second term reduces to that over \( Z_{l_{ki}} \) alone. The function \( F_\nu \) is the cdf of \( \max(\hat{\theta}_{l_{ki}} - Z_{l_{ki}}, \hat{\theta}_{l_{ki}}) \) under \( H_0: \hat{\theta}_{l_{ki}} = \hat{\theta}_{b_{ki}} = \ldots = \hat{\theta}_{l_{ki}}, \hat{\theta}_{b_{ki}} = \hat{\theta}_{b_{ki}} = \ldots = \hat{\theta}_{b_{ki}}, \hat{\theta}_{l_{ki}} = \hat{\theta}_{l_{ki}} = \ldots = \hat{\theta}_{l_{ki}}, \hat{\theta}_{b_{ki}} = \hat{\theta}_{b_{ki}} = \ldots = \hat{\theta}_{b_{ki}}, \hat{\theta}_{l_{ki}} = \hat{\theta}_{l_{ki}} = \ldots = \hat{\theta}_{l_{ki}} = 0 \) conditional on the adaptation decision (ii); and \( \alpha \) is the \((1 - \alpha) \times 100\)-th percentile of the standard normal distribution.

If, for some \( \nu^* \), \( \hat{\theta}_{l_{ki}} \leq c_2 \), then we reject \( H_{0, l_{ki}}: \hat{\theta}_{b_{ki}} \leq 0 \) and claim \( \delta^* = L^* \) if and only if

\[
w_1 \Phi^{-1}[F_\nu(Z_{l_{ki}})] + w_2 \Phi^{-1}[G_{\nu_{LL}}(\max(\hat{\theta}_{l_{ki}} - Z_{l_{ki}}, \hat{\theta}_{l_{ki}}))] > \alpha
\]

We note that the selection of the final second stage dose using
is consistent with the seamless quality of the design: information across both stages is used for this decision. Hypothesis testing, however, may not be done directly in terms of $\hat{S}_{e_{\text{pooled}}}$. Stagewise combination rules, as described in the next section, are needed to make valid inference.

The Limb-Leaf design is not limited to normal outcomes with known variance. Extensions to practical cases such as those of a large sample with consistently estimated nuisance parameters, and normally distributed data with unknown variance are presented in the Appendix. The design extends similarly to cases such as that of testing a location-shift hypothesis for continuous distributions using rank sum tests, and that of testing a difference in proportions for binary data using exact tests. These will be presented in later work.

3.2.3. Closed testing procedure and justification of the Limb-Leaf method

The search strategy outlined above entails selection over multiple dose levels, which calls for control of the familywise error rate (FWER) with respect to the collection of hypotheses $\{H_{d_j}: \theta_{d_j} \leq 0, d \in \mathcal{D}\}$, where FWER is the probability of rejecting any true null hypothesis. Control of the FWER for a given procedure means that FWER $\leq \alpha$ where the significance level $\alpha$ is prespecified. Furthermore, as argued by Tamhane et al. [18] among others, the appropriate form of familywise type 1 error control should be strong control, such that the FWER $\leq \alpha$ regardless of which hypotheses or how many hypotheses from the collection $\{H_{d_j}: \theta_{d_j} \leq 0, d \in \mathcal{D}\}$ are true. In other words we require that given a Limb-Leaf System $\mathcal{S}$, FWER(\(\theta_\theta\)) $\leq \alpha$ regardless of the shape of the underlying dose-response configuration \(\theta_\theta\). We emphasize that in a design that allows addition of doses as well as their promotion, control of the FWER only under the global null (weak control) does not imply the needed strong FWER control and is insufficient.

The closed testing procedure of Marcus et al. [13] guarantees strong control of the FWER with respect to a prespecified family of hypotheses which is closed under intersection. For a given set of hypotheses $H_{D_j} : j = 1, …, J$ the construction is as follows: For each subset $S$ of $\{1, …, J\}$ define the intersection hypothesis $H_{S.D} = \cap_{d \in S} H_{D_j}$ with corresponding level $\alpha$ test $\phi_S$ assumed to exist for each $S$. Then the closed testing principle requires that any hypothesis $H_{D_j}$ be rejected if and only if $H_{S.D}$ is rejected by $\phi_S$ for every set $S$ that contains $j$. The proof of strong control of the FWER is immediate: Let $S^*$ be the set of the indices of all true null hypotheses and assume $S^*$ is non-empty (or else there is nothing to prove). For a familywise error to be committed, $H_{S.D}$ must be rejected at level $\alpha$, which occurs with probability no greater than $\alpha$.

The Limb-Leaf procedure uses the following tests for hypotheses $H_0.D$. If $D$ contains any limb, then reject the null if and only if

$$Z_0 = w_1\phi^1[F_1(\max_{d \in D} Z_{1,d})) + w_2\phi^1[G_0(\max_{d \in D} Z_{2,d}))] \geq z_\alpha.$$  

Otherwise, reject the null if and only if

$$Z_0 = \phi^1[G_0(\max_{d \in D} Z_{2,d}))] \geq z_\alpha.$$  

Here $Z_{1,d}$ is the test statistic for $H_{0,D} : \theta_{d} \leq 0$ based on the 1st stage data from dose $d$ control if dose $d$ appears at Stage $i$, $i = 1, 2, F_p$ is the cdf of $\max_{\theta_\theta,D} Z_{1,d}$ under $\theta_\theta = 0, d \in D$, and $G_0$ is the cdf of $\max_{\theta_\theta,D} Z_{2,d}$ under $\theta_\theta = 0, d \in D$ conditional on the first stage data and thus on adaptation decisions based on them. If no dose $d \in D$ appears in the second stage, we set $Z_0 = -w_2$, as their corresponding doses have been deemed irrelevant after review of first stage data. It is emphasized that $w_1$ and $w_2$, $w_1^2 + w_2^2 = 1$, must be prespecified at the design stage in order for the test to be valid, and that the second form given is a special case of the first with $w_2 = 0$.

The following result provides a theoretical justification of the Limb-Leaf procedure. The proof is given in the Appendix.

**Theorem 1.** For any dose-response configuration $\theta_\theta$, the Limb-Leaf method specified in Section 3.2.2 yields strong control of the FWER at level $\alpha$.

Further flexibility in the Limb-Leaf design may be allowed. Selection of a dose other than the best performing at either stage does not undermine the validity of the familywise type 1 error rate control but may incur a power penalty that can be partly offset by a more complicated evaluation of all relevant intersection hypotheses. Dose selections can incorporate other information such as toxicity, cost or external data. Interim re-calculation of sample sizes, for instance to increase conditional power, would also be possible without undermining the validity of the trial. Finally, another potentially important type of flexibility would be in the definition of leaf doses; their exact dose levels or other aspects of their formulations could be finalized at the interim analysis without compromising the conditional sub-uniform distribution of their associated P-values and the procedure’s validity.

The validity of the Limb-Leaf design under other types of data is addressed by Theorem 2, also proved in the Appendix.

**Theorem 2.** The Limb-Leaf design extends to practical cases such as: 1) Normally distributed data with unknown variance $\sigma^2$, and 2) Large sample inference with consistently estimated nuisance parameters.

4. Power and optimization

For a given Limb-Leaf System $\mathcal{S}$ and a vector of effect thresholds $\delta$, we must prespecify the parameters $n_{1L}, n_{2L}, p_{1L}, p_{2L}, \mathcal{C} = (c_1, c_2)$, and weight $w_1$. To do this we minimize a risk-adjusted expected sample size under constraints on the power to identify and confirm an effect under specified alternatives.

4.1. Unfavorable configurations

Similar to Thall et al. [19], we choose certain unfavorable configurations (test configurations) in which we want to enforce adequate power. These are instances of locatable effects with respect to $\delta$ that are unfavorable to identification and confirmation of an effect on the correct target. One such configuration, the unfavorable limb effect configuration, $\theta_{\text{limb}}$, would satisfy the conditions:

$$\delta_{l_k} = \delta_{l_{i,1}} = \cdots = \delta_{l_{i,n_{l_k}}} = \delta_1 \quad \text{for} \quad k \neq K, \quad \delta_{l_{j,1}} = \delta_1, \quad \delta_{l_{j,1}} = \delta_{d_1}, \quad \delta_{l_{j,1}} = \delta_{d_2}, \quad \delta_{l_{j,1}} = \delta_2, \quad \delta_{l_{j,1}} = \delta_{d_3}, \quad \delta_{l_{j,1}} = \delta_3, \quad \text{for some limb dose } L_{d_i}.$$  

Analogously, we define an unfavorable leaf effect configuration $\theta_{\text{leaf}}$, as satisfying the conditions:

$$\delta_{l_{i,1}} = \delta_{l_{j,1}} = \cdots = \delta_{l_{j,n_{l_j}}} = \delta_1 \quad \text{for} \quad k \neq K, \quad \delta_{l_{j,1}} = \delta_2, \quad \delta_{l_{j,1}} = \delta_{d_1}, \quad \delta_{l_{j,1}} = \delta_{d_2}, \quad \delta_{l_{j,1}} = \delta_{d_3}, \quad \text{for some limb dose } L_{d_i} \text{ and leaf dose } L_{d_i}.\quad$$

The intuition of these choices is clear but a full characterization of the configurations that minimize the power would depend on assumptions concerning the construction of stagewise tests, combination rules, and interim dose selection rules. We do not address a full solution in this paper. Thus we call $\theta_{\text{limb}}$ and $\theta_{\text{leaf}}$ “unfavorable” rather than “least favorable” configurations.

4.2. Power and risk adjusted expected sample size

Let the risk-adjusted expected sample size, a form of Bayes’ risk, be defined as

$$E_r(N) = \pi_0E_0(N) + \theta_{\text{limb}}E_{\text{limb}}(N) + \theta_{\text{leaf}}E_{\text{leaf}}(N).$$  

where $E_0(N)$, $E_{\text{limb}}(N)$, and $E_{\text{leaf}}(N)$ denote the expected total sample sizes under the global null, unfavorable limb, and unfavorable leaf configurations, respectively, and $\pi_0$, $\pi_{\text{limb}}$ and $\pi_{\text{leaf}}$ denote their associated prior probabilities. Conservative values for early to mid-stage drug development would be $\pi_0 = 0.8$, $\pi_{\text{limb}} = 0.1$, and $\pi_{\text{leaf}} = 0.1$, for instance.

The design constants $n_{1L}, n_{2L}, p_{1L}, p_{2L}, \mathcal{C} = (c_1, c_2)$, and $w_1$ are selected to minimize $E_r(N)$, subject to power constraints

$$P_{\text{limb}}(\text{Confirm the treatment effect on limb } L_{d_i}) \geq 1 - \beta,$$

and

$$P_{\text{leaf}}(\text{Confirm the treatment effect on leaf } L_{d_i}) \geq 1 - \beta.$$
In general a nonlinear optimization procedure is required to find a possibly non-unique optimizer, or an approximate minimum can be found by a grid search over potential parameter values.

5. Simulation studies

We consider two schemas for comparison of a Limb-Leaf approach with a traditional TSE-style design. Both schemas are simple cases that could be used in practice. We compare performance characteristics over ranges of the δ1, δ2, δ3 parameters. In addition, robustness of the Limb-Leaf type design is considered in the following subsection. Without loss of generality we assume that the outcome of a patient on dose d is distributed as N(θδ,1), and the outcome of a control patient is distributed as N(0,1). All optimizations were conducted using computationally intensive grid-search algorithms. More efficient computational methods under development will be described in Section 6.

5.1. Schema A: A single limb

One of the simplest possible Limb-Leaf schemas (henceforth, “Schema A”) is as follows. There are 3 testing doses of interest, organized as: limb L1 and leaves l1,l1,1,2. Stage 1 compares L1 to control L0 where each arm would have sample size n1l. Depending on the results of this first stage comparison, we might: i) term the study; ii) continue the study with further recruitment to L0 and L1 of n2l1 subjects each in the second stage, or iii) continue with recruitment of n2l1 subjects each to L0 and L1 as well n2l2 subjects each to leaves l1,1 and l1,2. In case iii) we will allow selection of the optimal dose by comparison of overall sample means; although the inclusion of the first stage subjects in the sample mean of l1 is not completely unbiased, in practice this information may improve the probability of correct selection of the best dose. Evaluation of the results would be by the closed testing procedure utilizing pre-specified combination tests as described in Section 3. This could correspond, for instance, to a situation where investigators want the option to introduce interim modifications of their initial test dose L1 in case first stage results lead them to believe that either an increment or a decrement to the dose level might be necessary to optimize performance.

The corresponding TSE design would include L0 and l1,l1,1,2 as first stage doses, each with sample size n1l. If the first stage result of the best performing test dose δ1 exceeds the preset threshold y1, d would continue recruitment in the second stage along with control dose L0 with n2l1 subjects per arm. The efficacy of δ1 relative to L0 would then be determined by whether the overall sample mean of subjects assigned to δ1 exceeds the pre-specified y2.

The design constants to be set for the Limb-Leaf design are then n1l,n2l1,n2l2,n3l1, c = (c1,c2), as well as the combination rule weights w1 and w2. Their values will be determined to minimize E(N) defined in Section 4.2 with n0 = 0.8, n0limb = 0.1, and n0leaf = 0.1. Specifically, for given δ = (δ1,δ2,δ3), we let Θlimb be given by {θl1 = δ1, θl1 = δ2, θl2 = δ3}, and Θleaf be given by {θl1 = δ1, θl1 = δ2, θl2 = δ3}. Numerical optimization of the criterion E(N) subject to contraints of 0.9 power in both Θlimb and Θleaf configurations determines the design constants.

The corresponding TSE-type design will require the design constants n1l,n2l1,y1, and y2. Since the TSE-type design does not recognize a distinction between limb and leaf doses, both Θlimb and Θleaf may be expressed as ΘTSE, given by {θl1 = δ1, θl1 = δ2, θl2 = δ3}. The previous criterion, E(N) then reduces to 0.8E(N) + 0.2E0(N). The power constraints similarly reduce to the single restriction of 0.9 power in the ΘTSE configuration.

Below we present two tables, of optimized parameter values and performance characteristics for the two designs over ranges of δ = (δ1,δ2,δ3) for Schema A (Tables 1 and 2). Where this simple case assigns the same values of Θlimb,Θleaf, and ΘTSE for several values of δ, the optimization is unaffected by which one is chosen and the results are combined.

5.2. Schema B: Two limbs

A slightly more complex Limb-Leaf schema could include 6 doses of interest: limbs L1 and L2 each with two associated leaves, l1,l1,1,2 and l2,l2,1,2, respectively. Stage 1 includes l1,l1,1,2 and control L0 where each arm would have sample size n1l. Depending on the results of this first stage comparison, we might: i) term the study, ii) continue the study with further recruitment to L0 and L1, (the best performing limb in the first stage) of n2l1 subjects each in the second stage, or iii) continue with recruitment of n2l2 subjects each to L0, L1, and L2 as well n2l2 subjects each to leaves l1,1 and l1,2. In case iii) we again allow final selection of the optimal dose by comparison of overall sample means, possibly across both stages. Evaluation of the results would be by the closed testing procedure utilizing pre-specified combination tests as described in Section 3. This could correspond, for instance, to a situation where investigators have a prior belief that one of two chosen doses may provide sufficient efficacy, however, they also want the option to further explore around the dose level that appears promising in order to fine tune towards an optimal dose.

The corresponding TSE design includes L0,l1,l1,1,2,l2,l2,1,2 as first stage doses, each with sample size n1l. If the first stage result of the best performing test dose δ1 exceeds the preset threshold y1, d would continue recruitment in the second stage along with control dose L0 with n2l1 subjects per arm. The efficacy of δ1 relative to L0 would then be determined by whether the overall sample mean of subjects assigned to δ1 exceeds the pre-specified y2.

The design constants to be set for the Limb-Leaf design are as before n1l,n2l1,n2l2,n3l1, c = (c1,c2), and the weights w1 and w2. We minimize E(N) defined in Section 4.2 with n0 = 0.8, n0limb = 0.1, and n0leaf = 0.1. In this case, for given δ = (δ1,δ2,δ3), we let Θlimb be given by {θl1 = δ1, θl1 = δ2, θl2 = δ3}, and Θleaf be given by {θl1 = δ1, θl1 = δ2, θl2 = δ3}. Numerical optimization of the criterion E(N) subject to contraints of 0.9 power in both Θlimb and Θleaf configurations determines design constants.

The corresponding TSE-type design will require the design constants n1l,n2l1,y1, and y2. Since the TSE-type design does not recognize a distinction between limb and leaf doses, both Θlimb and Θleaf may be expressed as ΘTSE, given by {θl1 = δ1, θl1 = δ2, θl2 = δ3}. The previous criterion, E(N) then reduces to 0.8E(N) + 0.2E0(N). The power constraints similarly reduce to the single restriction of 0.9 power in the ΘTSE configuration. Numerical optimization subject to the power constraint then determines values of

<table>
<thead>
<tr>
<th>Table 1</th>
<th>TSE design constants under Schema A.</th>
</tr>
</thead>
<tbody>
<tr>
<td>δ1</td>
<td>δ2</td>
</tr>
<tr>
<td>0.0</td>
<td>0.2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
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<tr>
<td>0.6</td>
<td>0.8</td>
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</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Limb-Leaf design constants under Schema A.</th>
</tr>
</thead>
<tbody>
<tr>
<td>δ1</td>
<td>δ2</td>
</tr>
<tr>
<td>0.0</td>
<td>0.2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8</td>
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Table 3

TSE design constants under Schema B.

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
<th>$\delta_3$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_{L1}$</th>
<th>$n_{L2}$</th>
<th>$n_{TSE}$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$E_1(N_{TSE})$</th>
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<tbody>
<tr>
<td>0.0</td>
<td>0.2</td>
<td>1.0</td>
<td>18</td>
<td>12</td>
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<td>0.550</td>
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<td></td>
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<td>0.4</td>
<td>1.0</td>
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<tr>
<td>0.0</td>
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<td>1.0</td>
<td>36</td>
<td>2</td>
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<td>0.400</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.0</td>
<td>122</td>
<td>2</td>
<td>0.650</td>
<td>0.575</td>
<td>854.8</td>
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<td></td>
<td></td>
<td></td>
</tr>
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<td>0.525</td>
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<tr>
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<td>1.0</td>
<td>36</td>
<td>2</td>
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<td>0.375</td>
<td>292.9</td>
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</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>1.0</td>
<td>124</td>
<td>2</td>
<td>0.650</td>
<td>0.385</td>
<td>868.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>0.4</td>
<td>0.6</td>
<td>1.0</td>
<td>38</td>
<td>4</td>
<td>0.550</td>
<td>0.150</td>
<td>267.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>1.0</td>
<td>122</td>
<td>4</td>
<td>0.675</td>
<td>0.325</td>
<td>855.6</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
<td>122</td>
<td>4</td>
<td>0.675</td>
<td>0.150</td>
<td>855.6</td>
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Table 4

Limb-Leaf design constants under Schema B.

<table>
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<tr>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
<th>$\delta_3$</th>
<th>$n_{L1}$</th>
<th>$n_{L2}$</th>
<th>$n_{TSE}$</th>
<th>$y_{L1}$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$w_1$</th>
<th>$E_1(N_{TSE})$</th>
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<tbody>
<tr>
<td>0.0</td>
<td>0.2</td>
<td>1.0</td>
<td>85</td>
<td>52</td>
<td>42</td>
<td>22</td>
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<td>0.675</td>
<td>0.050</td>
<td>426.0</td>
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<td>0.4</td>
<td>1.0</td>
<td>33</td>
<td>40</td>
<td>37</td>
<td>14</td>
<td>-0.050</td>
<td>1.700</td>
<td>0.050</td>
<td>216.3</td>
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<tr>
<td>0.0</td>
<td>0.6</td>
<td>1.0</td>
<td>28</td>
<td>36</td>
<td>48</td>
<td>31</td>
<td>0.125</td>
<td>1.350</td>
<td>0.125</td>
<td>178.7</td>
</tr>
<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.0</td>
<td>37</td>
<td>112</td>
<td>156</td>
<td>65</td>
<td>0.350</td>
<td>1.750</td>
<td>0.275</td>
<td>266.5</td>
</tr>
<tr>
<td>0.0</td>
<td>0.4</td>
<td>1.0</td>
<td>97</td>
<td>40</td>
<td>59</td>
<td>37</td>
<td>0.125</td>
<td>0.825</td>
<td>0.200</td>
<td>367.3</td>
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<tr>
<td>0.0</td>
<td>0.6</td>
<td>1.0</td>
<td>44</td>
<td>51</td>
<td>55</td>
<td>44</td>
<td>0.200</td>
<td>1.300</td>
<td>0.1125</td>
<td>218.7</td>
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<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.0</td>
<td>39</td>
<td>111</td>
<td>148</td>
<td>12</td>
<td>0.325</td>
<td>1.575</td>
<td>0.175</td>
<td>275.5</td>
</tr>
<tr>
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<td>0.6</td>
<td>1.0</td>
<td>90</td>
<td>54</td>
<td>88</td>
<td>36</td>
<td>0.200</td>
<td>1.050</td>
<td>0.325</td>
<td>352.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.0</td>
<td>50</td>
<td>108</td>
<td>178</td>
<td>6</td>
<td>0.350</td>
<td>1.625</td>
<td>0.075</td>
<td>294.5</td>
</tr>
<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.0</td>
<td>143</td>
<td>130</td>
<td>188</td>
<td>28</td>
<td>-0.450</td>
<td>1.300</td>
<td>0.275</td>
<td>1064.7</td>
</tr>
</tbody>
</table>

Interpretation

The ratio of risk adjusted expected total sample size between the Limb-Leaf and TSE designs is less than 1, favoring the Limb-Leaf design. This indicates that the Limb-Leaf design is more efficient than the TSE design, especially for certain values of the parameters $\delta_1$, $\delta_2$, and $\delta_3$. The results are robust to variation in the parameters, suggesting that the Limb-Leaf design is a reliable choice for the given scenarios.

5.3. Interpretation

The interpretation of the comparison between designs is aided by the following tables. The key figure is $E_1(N_{TSE})/E_1(N_{TSE})$, the ratio of risk adjusted expected total sample size. This ratio is less than 1, indicating that the Limb-Leaf design is more efficient than the TSE design.

5.4. Robustness against distortions for selected configurations

We conduct a sensitivity analysis to assess the robustness of the Limb-Leaf design against distortions in the parameters $\delta_1$, $\delta_2$, and $\delta_3$. The results show that the Limb-Leaf design is robust to variation in these parameters, maintaining its efficiency and power across a wide range of values. This suggests that the Limb-Leaf design is a reliable choice for clinical trials, as it performs well under different scenarios.

The selection of design constants appropriate for the locatable effect assumption

In practice, we would expect to use a maximum of three limbs and no more than two leaves per limb. The ratio of risk adjusted expected total sample size between the Limb-Leaf and TSE designs is less than 1, favoring the Limb-Leaf design. This indicates that the Limb-Leaf design is more efficient than the TSE design, especially for certain values of the parameters $\delta_1$, $\delta_2$, and $\delta_3$. The results are robust to variation in the parameters, suggesting that the Limb-Leaf design is a reliable choice for the given scenarios.

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In practice, we would expect to use a maximum of three limbs and no more than two leaves per limb. The ratio of risk adjusted expected total sample size between the Limb-Leaf and TSE designs is less than 1, favoring the Limb-Leaf design. This indicates that the Limb-Leaf design is more efficient than the TSE design, especially for certain values of the parameters $\delta_1$, $\delta_2$, and $\delta_3$. The results are robust to variation in the parameters, suggesting that the Limb-Leaf design is a reliable choice for the given scenarios.
assumed value of $\delta = (0.0, 0.8, 1.0)$, a $\delta_3^*$ value corresponding to a 2.5% reduction in $\delta_3$ is associated with a 5% reduction in power while a 5% reduction in $\delta_3$ is associated with a 10% reduction in power. On the other hand, values of $\delta_3^*$ greater than $\delta_3$ are not associated with negative impacts on power. For design purposes, an initial estimate of $\delta_3$ on the low end of its assumed range would be a conservative choice.

6. Discussion

The proposed Limb-Leaf procedure is a structured exploration strategy that builds on earlier adaptive designs such as those proposed in Bauer and Keiser [1] and Bauer and Köhne [2]. It is a generalization of the design by Thall et al. [19] targeted to improve performance under a possibly non-monotonic dose-response relationship. These have become more relevant in recent years as new therapies often entail increasingly complex mechanisms of action and combination therapies may not allow a clear ordering among their dosage levels. When $\delta = (\delta_1, \delta_2, \delta_3)$ is chosen appropriately, for example when $\delta_2$ is at least 60% of $\delta_3$, the saving in total sample size using a Limb-Leaf design is substantial compared with the TSE design. Limitations on the proper set-

### Table 7

Allowed ranges of $B' = (\delta_1', \delta_2', \delta_3')$ for the Limb-Leaf designs under schema A.

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
<th>$\delta_3$</th>
<th>$\delta_1'$</th>
<th>$\delta_2'$</th>
<th>$\delta_3'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.6</td>
<td>1.0</td>
<td>$(-\infty, 0.60)$</td>
<td>[0.29, 0.67]</td>
<td>(0.95, $\infty$)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>1.0</td>
<td>$(-\infty, 0.80)$</td>
<td>[0.60, 0.83]</td>
<td>(0.97, $\infty$)</td>
</tr>
</tbody>
</table>

### Table 8

Allowed ranges of $B' = (\delta_1', \delta_2', \delta_3')$ for the Limb-Leaf designs under schema B.

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
<th>$\delta_3$</th>
<th>$\delta_1'$</th>
<th>$\delta_2'$</th>
<th>$\delta_3'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.6</td>
<td>1.0</td>
<td>$[-0.58, 0.18]$</td>
<td>[0.47, 0.69]</td>
<td>(0.95, $\infty$)</td>
</tr>
<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.0</td>
<td>$[-0.50, 0.43]$</td>
<td>[0.61, 0.83]</td>
<td>(0.98, $\infty$)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.6</td>
<td>1.0</td>
<td>$[-0.48, 0.33]$</td>
<td>[0.50, 0.73]</td>
<td>(0.91, $\infty$)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>1.0</td>
<td>$[-0.49, 0.44]$</td>
<td>[0.59, 0.83]</td>
<td>(0.98, $\infty$)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
<td>1.0</td>
<td>$[-0.43, 0.51]$</td>
<td>[0.65, 0.84]</td>
<td>(0.97, $\infty$)</td>
</tr>
</tbody>
</table>

inverse normal rule used in this paper for stage-wise $Z$-values are possible. Fisher’s combination rule, for instance, may be a robust choice if some doses are expected to perform especially poorly and one wants to limit their impact on tests of effect of unrelated leaf doses. Further study on combination rules for robustness and efficiency is warranted. Along with different combination rules, different rules for stochastic curtailment may be employed. The efficiency gain of group sequential continuity [14] as well as dropping poorly performing treatment arms at potentially earlier interim points is worth investigation as is the comparison of relative benefit in a TSE-style approach.

To accomplish point and interval estimation in the context of Limb-Leaf designs existing methods based on bias adjusted estimators [17], test inversion to form confidence regions, and median unbiased point estimators are applicable [4,6,8,11]. The specific implementation of these methods and comparison of their performance will be treated in future research.

A final issue is the computational complexity of parameter optimization in a Limb-Leaf type design. An R program developed by Mr. X. Lu using simulated annealing for nonlinear optimization appears to offer substantially improved speed over a basic grid search method such that a design can be optimized in less than one hour on a Windows desktop computer with 3 GHz processing speed. A preliminary version of this software is available from the authors upon request. A full version is planned for public release as an R package.

### Acknowledgments

We thank Mr X. Lu for offering his expertise in statistical programming.

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### Appendix A

**Proof of Proposition 1.** For simplicity assume that the effects of no two doses are identical. By renumbering the limbs as necessary, we may take the limb effects $\theta_{ik}$, $k = 1, \ldots, K$, as forming an increasing sequence. Choose $\delta_1$ such that $\theta_{ik_{\delta_1}} < \theta_{i(k-1)}$. Then, the effects of all leaf doses associated with $\theta_{ik}$ may be written as an increasing sequence $\theta_{ik_{\delta_1}} > \ldots > \theta_{ik_{\delta_3}}$

There are then two subcases. The first applies if $\theta_{ik_{\delta_3}} > \theta_{ik}$. Choose $\delta_2$ such that $\delta_1 < \delta_2 < \theta_{ik}$. Let $l^h = \min\{i: \theta_{ik}, \theta_{ik_{\delta_2}}\}$, the lowest leaf dose whose effect exceeds that of $l^h$. We may then choose $\delta_3$ such that $\theta_{ik_{\delta_3}} < \theta_{ik_{\delta_2}}$. The conditions for a locatable effect with respect to $\delta = (\delta_1, \delta_2, \delta_3)$ are then satisfied.

If $\theta_{ik_{\delta_3}} < \theta_{ik}$ then let $\gamma = \max\{\delta_1, \theta_{ik_{\delta_3}}, \theta_{ik_{\delta_2}}\}$. Choose $\delta_2$ and $\delta_3$ such that $\gamma < \delta_2 < \delta_3 < \theta_{ik}$. The conditions for a locatable effect with respect to $\delta = (\delta_1, \delta_2, \delta_3)$ are similarly satisfied.

This construction may be modified to produce other solutions and to allow perfect ties between dose effects although the notation becomes more complicated.
Proof of Theorem 1. We demonstrate that when \( H_0;\theta^* \) is rejected, so are all the intersection hypotheses that contain it. Given data, we use \( z_{i,d} \) to denote the observed \( Z_{i,d} \) and relabel doses as necessary for convenience of notation.

Suppose \( c_1 < \hat{\theta};_{1,2} \leq c_2 \) and \( d^* = L_k^* \). A composite hypothesis including \( H_{0,k} \) may be written as \( H_{0,D} \) where

\[
D = \{ L_k^* \} \cup \{ L_{k,1}, \ldots, L_{k,m}^* \} \cup \{ L_{k,1}, \ldots, L_{k,m} \} \cup \ldots \cup \{ L_{k,1}, \ldots, L_{k,m}^* \}.
\]

Here \( L_j \neq L_k^* \), \( K \leq K - 1 \), and \( 0 \leq m_j \leq m_{j,*} \). For \( K' \) or \( m_{j,*} \) equal to zero, the corresponding terms in braces are counted as empty. Let \( k^* \in \{0, m_{j,*} \} \) denote the number of leaves in \( D \) that were utilized (randomized to) after the interim decision. Then the relations

\[
z_D = w_1 \Phi^{-1} \{ F_{1+k} \{ z_{2+k}, \ldots, z_{2+k,m} \} \} + w_2 \Phi^{-1} \{ G_{\alpha_k} \{ \max (z_{2+k}, z_{2+k,m}) \} \}
\]

where the second inequality is assumed from the rejection of \( H_0;\theta^* \). We consider \( D \) in three cases.

Case 1. If \( D \) contains \( L_k^* \), using the same notations as above,

\[
z_D = w_1 \Phi^{-1} \{ F_{1+k} \{ z_{2+k} \} \} + w_2 \Phi^{-1} \{ G_{\alpha_k} \{ \max (z_{2+k}, z_{2+k,m}) \} \}
\]

where the last inequality is a rejection condition for \( H_{0,k^*}. \)

Case 2. If \( D \) does not contain \( L_k^* \), but contains some other limb, we write

\[
D = \{ L_{k,1}, \ldots, L_{k,m} \} \cup \{ L_{k,1}, \ldots, L_{k,m}^* \} \cup \ldots \cup \{ L_{k,1}, \ldots, L_{k,m}^* \}
\]

using the previous notations. Then,

\[
z_D = w_1 \Phi^{-1} \{ F_k \{ \max (z_{2+k} \ldots, z_{2+k,m}) \} \} + w_2 \Phi^{-1} \{ G_{\alpha_k} \{ \max (z_{2+k}, z_{2+k,m}) \} \}
\]

where the second inequality is assumed from the rejection of \( H_0;\theta^* \). Finally, suppose \( \hat{\theta}_{1,2} \geq c_2 \). We write

\[
D = \{ L_k^* \} \cup \{ L_{k,1}, \ldots, L_{k,m} \} \cup \{ L_{k,1}, \ldots, L_{k,m}^* \} \cup \ldots \cup \{ L_{k,1}, \ldots, L_{k,m}^* \}
\]

where the second inequality is assumed from the rejection of \( H_0;\theta^* \), \( H_0;\theta^* \) is rejected. This concludes the proof of the theorem.

Proof of Theorem 2.

1. For normally distributed outcomes with unknown \( \sigma^2 \), let \( Z_{1,k} = (X_{1,k} - \bar{X}_{1,k}) / \sqrt{\hat{\sigma}^2_{1,k} / n_k} \) denote the first stage test statistic of the effect of dose \( k \), \( k = 1, \ldots, K_m \) where \( \hat{\sigma}^2_{1,k} \) is the pooled estimate of \( \sigma^2 \) in the control arm and arm \( L_k \) from stage 1. Instead of being normal with known variance, each \( Z_{1,k} \) now follows a t-distribution with \( 2n_{1,k} - 1 \) df under its associated null hypothesis of no treatment effect.

Similarly, let \( Z_{2,k} = (X_{2,k} - \bar{X}_{2,k}) / \sqrt{\hat{\sigma}^2_{2,k} / n_k} \) or \( (X_{2,k} - \bar{X}_{2,k}) / \sqrt{\hat{\sigma}^2_{2,k} / n_k} \) as required denote the second stage test statistic for the effect of dose \( L_k \), \( k = 1, \ldots, K_m \) where \( \hat{\sigma}^2_{2,k} \) is the pooled estimate of \( \sigma^2 \) in the control arm and arm \( L_k \) in stage 2. \( Z_{2,k} \) follows a t-distribution with \( 2n_{2,k} - 1 \) df under its associated null hypothesis. We may also define \( Z_{2,0} = (X_{2,0} - \bar{X}_{2,0}) / \sqrt{\hat{\sigma}^2_{2,0} / n_k} \), \( k = 1, \ldots, m^* \), where \( \hat{\sigma}^2_{2,k} \) is the pooled estimate of \( \sigma^2 \) in the control arm and arm \( L_k \) in stage 2, such that under the associated null hypothesis it follows a t-distribution with \( n_{2,k} + n_{2,k} - 2 \) df for prespecified values of \( n_{2,k}, n_{2,k}, n_{2,k} \) and \( n_{2,k} \), the functions \( F_k, G_{\alpha_k}, G_{\alpha_k^*} \) are then well defined under each adaptation; they can be calculated or approximated by Monte-Carlo simulation from the reference case of unit variance since the parameter \( \sigma^2 \) has been removed by invariance. Subject only to these changes in interpretation, the design of Section 3.2.2 applies and the proof of Theorem 1 applies to guarantee
control of the FWER at level $\alpha$.

The Satterthwaite approximation may be used as usual to accommodate unequal variances.

2. A general method to draw inferences about parameters of interest in large samples is based on the calculation of the score statistic and observed Fisher information. The method is valid for general endpoints under regularity conditions satisfied, for instance, when the response distribution belongs to an exponential family; nuisance parameters are accommodated through use of consistent estimators. Details are given by Stallard and Todd [16].

Theoretical justification is given by Whitehead [21].

Let $d_{\theta}$ be a measure of the superiority of treatment $d$ relative to control with $d_{\theta} > 0$ indicating superiority, and let $S_{d,\theta}$ and $V_{d,\theta}$ denote the efficient score and observed Fisher Information for $d_{\theta}$ at stages $i = 1, 2$, (evaluated at $d_{\theta} = 0$). It may be shown that the asymptotic joint distribution of the

$Z_{1,\theta} = \frac{s_{1,\theta}}{\sqrt{V_{1,\theta}}}, \ldots, Z_{2,\theta} = \frac{s_{2,\theta}}{\sqrt{V_{2,\theta}}}$

is multivariate normal with mean proportional to the vector of assumed effect sizes and fixed covariance matrix.

Similarly, given any adaptation decision applied in the second stage, $Z_{2,\theta'} = \frac{s_{2,\theta'}}{\sqrt{V_{2,\theta'}}}, Z_{2,\theta'1} = \frac{s_{2,\theta'1}}{\sqrt{V_{2,\theta'1}}}, \ldots, Z_{2,\theta'k} = \frac{s_{2,\theta'k}}{\sqrt{V_{2,\theta'k}}}$ have an asymptotic distribution with mean proportional to the vector of assumed effect sizes and fixed covariance matrix. Stagewise effect estimates may be calculated as

$\hat{\theta}_{1,\theta} = \frac{Z_{1,\theta}}{V_{1,\theta}}, \frac{Z_{2,\theta'}}{V_{2,\theta'}}$, and $\hat{\theta}_{2,\theta'1} = \frac{Z_{2,\theta'1}}{V_{2,\theta'1}}, \ldots, \hat{\theta}_{2,\theta'k} = \frac{Z_{2,\theta'k}}{V_{2,\theta'k}}, l = 1, \ldots, m_{\theta'}$.

It follows that the plan of a Limb-Leaf Design of Section 3.2.2 can be implemented with the above new meanings for the symbols $\hat{\theta}_{1,\theta}, \hat{\theta}_{2,\theta'},$ and $\hat{\theta}_{2,\theta'1}, \ldots, \hat{\theta}_{2,\theta'k},$ as well as $Z_{1,\theta'}, \ldots, Z_{2,\theta'}, Z_{2,\theta'1}, \ldots, Z_{2,\theta'k}$, the functions $F_{\theta'}, G_{m_\theta'}, G_{limb+m_{\theta'}}$ are then well defined. Subject only to these changes in interpretation, the design of Section 3.2.2 applies and the proof of Theorem 1 shows that the FWER is controlled approximately at level $\alpha$ in large samples.

References


