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Answer to an open problem on mean residual life ordering of parallel systems under multiple-outlier exponential models

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ABSTRACT

In a comprehensive survey on stochastic comparison, Balakrishnan and Zhao (2013) proposed an open problem on mean residual life ordering between two parallel systems under multiple-outlier exponential models. This paper provides an affirmative answer to this open problem.

1. Introduction

Order statistics is closely related to the lifetime of a k-out-of-n system. Stochastic comparison of the largest order statistic which characterizes the lifetime of a parallel system is an active research area where many important results have been established.

To facilitate the discussion, we introduce some terminologies whose detailed definitions are given in next section. Let $X_i, i = 1, \ldots, n$, be a set of independent exponential random variables with hazard rates λ_i, and $Y_i, i = 1, \ldots, n$, be another independent set of independent exponential random variables with hazard rates μ_i. By symmetry, we assume that the hazard rate vectors are sorted in an ascending order. That is, $\lambda_1 \le \cdots \le \lambda_n$ and $\mu_1 \le \cdots \le \mu_n$. Let $X_{n:n} = \max\{X_1, \ldots, X_n\}$, and $Y_{n:n} = \max\{Y_1, \ldots, Y_n\}$. Several orderings of hazard rate vector are considered. Specifically, $\overset{m}{\succ}$ denotes the majorization order; $\overset{rm}{\succ}$ the reciprocal majorization order; $\overset{p}{\succ}$ the p-larger order; $\overset{w}{\succ}$ the weak majorization order. For system lifetime stochastic orderings, $\ge_{st}$ denotes the usual stochastic order, $\ge_{hr}$ the hazard rate order, $\ge_{lr}$ the likelihood ratio order, and $\ge_{mrl}$ the mean-residual life order.

The research on stochastic comparison for the largest statistics has been very active. See Kochar (2012) and Balakrishnan and Zhao (2013) for a comprehensive review of the recent progresses in this area. For a more in-depth exposition of the important results, see, for instance, Proschan and Sethuraman (1976), Kochar and Rojo (1996), Dykstra et al. (1997), Khaledi and Kochar (2000), Kochar and Xu (2007), Zhao and Balakrishnan (2011a), Yan et al. (2013), and Balakrishnan and Zhao (2013).

Specifically, for $n = 2$, Dykstra et al. (1997) proved that

$$(\lambda_1, \lambda_2) \overset{m}{\succ} (\mu_1, \mu_2) \Longrightarrow X_{2:2} \ge_{lr} Y_{2:2}. \tag{1.1}$$

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Under condition

$$0 < \lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2, \tag{1.2}$$

Joo and Mi (2010) showed that

$$(\lambda_1, \lambda_2) \overset{w}{\succ} (\mu_1, \mu_2) \Longrightarrow X_{2:2} \ge_{hr} Y_{2:2}.$$

Zhao and Balakrishnan (2011b) enhanced the above result to

$$(\lambda_1, \lambda_2) \overset{w}{\succ} (\mu_1, \mu_2) \Longleftrightarrow X_{2:2} \ge_{lr} Y_{2:2}, \tag{1.3}$$

and in addition they proved that

$$(\lambda_1, \lambda_2) \overset{p}{\succ} (\mu_1, \mu_2) \Longleftrightarrow X_{2:2} \ge_{hr} Y_{2:2}. \tag{1.4}$$

Zhao and Balakrishnan (2012b) revealed that, under condition (1.2),

$$(\lambda_1, \lambda_2) \overset{p}{\succ} (\mu_1, \mu_2) \Longrightarrow X_{2:2} \ge_{disp} Y_{2:2}. \tag{1.5}$$

For MRL ordering, Zhao and Balakrishnan (2011a) showed that, under condition (1.2),

$$(\lambda_1, \lambda_2) \overset{rm}{\succ} (\mu_1, \mu_2) \Longrightarrow X_{2:2} \ge_{mrl} Y_{2:2}. \tag{1.6}$$

The stochastic comparison of $X_{n:n}$ and $Y_{n:n}$ for a general n and a general hazard rate vector is challenging. Instead, researchers focus on the so-called multiple-outlier exponential model which consists of p independent exponential components of hazard rate λ_1 and $q = n - p$ independent exponential components of hazard rate λ_2, $\lambda_1 \le \lambda_2$. More specifically, if, for $i = 1, \ldots, p$, X_i follow an exponential distribution of hazard rate λ_1, and for $j = p + 1, \ldots, n$, X_j follow an exponential distribution of hazard rate λ_2, we then say that $X_1, \ldots, X_n$ follow a multiple-outlier exponential model of type (p, q) with hazard rate vector (λ_1, λ_2).

Thus far, results (1.1) through (1.5) have been successfully extended to the general n case under the multiple-outlier models except for result (1.6). Specifically, Torrado and Kochar (2015) extended result (1.1) to the multiple-outlier Weibull models. Results (1.3) and (1.4) have been extended to the multiple-outlier models in Zhao and Balakrishnan (2011b), and result (1.5) in a multiple-outlier model version has been proved in Balakrishnan and Zhao (2013). It is natural to ask whether result (1.6) can be extended to the multiple-outlier models. Actually, this is one of the open problems proposed in Balakrishnan and Zhao (2013).

In this paper, we provide an affirmative solution to this open problem. The main result presented in this paper can be stated as, under the condition (1.2),

$$(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2) \overset{rm}{\succ} (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2) \Longrightarrow X_{n:n} \ge_{mrl} Y_{n:n}.$$

The paper is organized as follows. Section 2 gives the required notations and definitions. Section 3 presents the proof of the main result. A brief discussion is provided in Section 4.

2. Notations and definitions

This section presents some notations and definitions of stochastic orders. For more details on stochastic orders, see Shaked and Shanthikumar (2007) and Balakrishnan and Zhao (2013).

Let X be a nonnegative continuous random variable with distribution function $F_X(t)$, survival function $\bar{F}_X(t) = 1 - F_X(t)$, and density function $f_X(t)$. The hazard function and the reversed hazard function of X are defined as $\lambda_X(t) = f_X(t)/\bar{F}_X(t)$ and $r_X(t) = f_X(t)/F_X(t)$, respectively. For two nonnegative continuous random variables X and Y, we say that X is larger than Y in the usual stochastic order (denoted by $X \ge_{st} Y$), if $\bar{F}_X(t) \ge \bar{F}_Y(t)$; X is larger than Y in the hazard rate order (denoted by $X \ge_{hr} Y$), if $\lambda_X(t) \ge \lambda_Y(t)$; X is larger than Y in the reversed hazard rate order (denoted by $X \ge_{rh} Y$), if $r_X(t) \ge r_Y(t)$; X is larger than Y in the likelihood ratio order (denoted by $X \ge_{lr} Y$), if the ratio $f_X(t)/f_Y(t)$ is increasing in t; X is larger than Y in the mean residual life (MRL) order (denoted by $X \ge_{mrl} Y$), if $E(X - t|X > t) \ge E(Y - t|Y > t)$; X is said to be less dispersed than Y (denoted by $X \le_{disp} Y$) if $F^{-1}(v) - F^{-1}(u) \le G^{-1}(v) - G^{-1}(u)$ for $0 \le u \le v \le 1$, where F^{-1} and G^{-1} are the right continuous inverses of the distribution functions F and G of X and Y, respectively.

Given two vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ with increasing elements, vector $\boldsymbol{a}$ is said to majorize vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \overset{m}{\succ} \boldsymbol{b}$) if, $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, and $\sum_{i=1}^k a_i \le \sum_{i=1}^k b_i$, for $k = 1, \ldots, n-1$; vector $\boldsymbol{a}$ is said to weakly majorize vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \overset{w}{\succ} \boldsymbol{b}$) if, $\sum_{i=1}^k a_i \le \sum_{i=1}^k b_i$, for $k = 1, \ldots, n$; vector $\boldsymbol{a}$ is said to be p-larger than vector $\boldsymbol{b}$, introduced by Bon and Păltănea (1999) and denoted as $\boldsymbol{a} \overset{p}{\succ} \boldsymbol{b}$, if $\prod_{i=1}^k a_i \le \prod_{i=1}^k b_i$, for $k = 1, \ldots, n$; vector $\boldsymbol{a}$ is said to reciprocally majorize vector $\boldsymbol{b}$, introduced by Zhao and Balakrishnan (2009) and denoted as $\boldsymbol{a} \overset{rm}{\succ} \boldsymbol{b}$, if $\sum_{i=1}^j 1/a_i \ge \sum_{i=1}^j 1/b_i$, $j = 1, \ldots, n$.

The following lemma will be used in the proof of the main theorem.

Lemma 2.1. *For any nonnegative integers p, q, $p+q \geq 1$, and for $k \geq 1$, function*

$$H(p, q, k; x) = \frac{1-(1-e^{-x})^p(1-e^{-kx})^q}{xe^{-x}(1-e^{-x})^{p-1}(1-e^{-kx})^q},$$

is a decreasing function on $x > 0$.

Proof. Rewrite $xH(p, q, k; x)$ as

$$xH(p, q, k; x) = \frac{e^{px+kqx}-(e^x-1)^p(e^{kx}-1)^q}{(e^x-1)^{p-1}(e^{kx}-1)^q}.$$

Denote $u = e^x - 1$, $v = e^{kx} - 1$. Then, for $x > 0$, $u, v > 0$, and

$$\begin{aligned} xH(p, q, k; x) &= \frac{(1+u)^p(1+v)^q - u^p v^q}{u^{p-1}v^q} \\ &= \sum_{i,j}^{p,\,q} \binom{p}{i}\binom{q}{j} u^{-[(p-1)-i]} v^{-(q-j)} \rho_{ij} \\ &= \sum_{i,j}^{p-1,\,q} \binom{p}{i}\binom{q}{j} u^{-[(p-1)-i]} v^{-(q-j)} + \sum_{j}^{q-1} \binom{q}{j} u v^{-(q-j)}, \end{aligned}$$

where the indicator function ρ_{ij} is 0 when $(i, j) = (p, q)$ and 1 otherwise. For $i \leq (p-1)$, the term $u^{-[(p-1)-i]}v^{-(q-j)}$ is decreasing in x. For $i = p$, the term $uv^{-(q-j)} = uv^{-1}v^{-(q-j)+1}$ is also decreasing, since $-(q-j)+1 \leq 0$, and $uv^{-1} = e^x-1/e^{kx}-1$ is decreasing when $k \geq 1$. Hence, $xH(p, q, k; x)$ is decreasing, and since $1/x$ is also decreasing, thus their product $H(p, q, k; x)$ is also decreasing over $x > 0$. □

3. Main result and its proof

Theorem 3.1. *Suppose $X_1, \ldots, X_n$ follow a multiple-outlier exponential model of type (p, q) with hazard rate vector (λ_1, λ_2), and $Y_1, \ldots, Y_n$ follow a multiple-outlier exponential model of type (p, q) with hazard rate vector (μ_1, μ_2). Then, under the condition (1.2), we have,*

$$(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2) \overset{rm}{\succ} (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2) \Longrightarrow X_{n:n} \geq_{mrl} Y_{n:n}.$$

Proof. Denote $X_{n:n}$'s distribution function as $F(p, q, \lambda_1, \lambda_2; t) = (1-e^{-\lambda_1 t})^p(1-e^{-\lambda_2 t})^q$, and its density function as $f(p, q, \lambda_1, \lambda_2; t)$. By the fact that for any nonnegative continuous random variable W, $EW = \int_0^\infty P(W > t)dt$, and that the conditional density function of $X_{n:n}|(X_{n:n} > t)$ is $\frac{f(p,q,\lambda_1,\lambda_2;u)}{1-F(p,q,\lambda_1,\lambda_2;t)}$, we conclude that the MRL function $X_{n:n}$ is

$$\begin{aligned} \varphi_{\boldsymbol{\lambda}}(t) &= E(X_{n:n} - t | X_{n:n} > t) \\ &= \int_t^\infty P(X_{n:n} > u | X_{n:n} > t) du \\ &= \int_t^\infty \int_u^\infty \frac{f(p, q, \lambda_1, \lambda_2; w)}{1-F(p, q, \lambda_1, \lambda_2; t)} dw du \\ &= \int_t^\infty \frac{1-F(p, q, \lambda_1, \lambda_2; u)}{1-F(p, q, \lambda_1, \lambda_2; t)} du \\ &= \int_t^\infty \{1-F(p, q, \lambda_1, \lambda_2; u)\} du \Big/ \{1-F(p, q, \lambda_1, \lambda_2; t)\}. \end{aligned}$$

Similarly, the MRL function of $Y_{n:n}$ is $\varphi_{\boldsymbol{\mu}}(t)$. Here $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2)$.

For $x_2 > x_1 > 0$, consider function

$$\Phi(x_1, x_2) = \int_1^\infty \{1-F(p, q, x_1, x_2; u)\} du \Big/ \{1-F(p, q, x_1, x_2; 1)\}.$$

By a change of variable with $w = u/t$ in the integral,

$$\varphi_{\boldsymbol{\lambda}}(t) = \frac{t\int_1^\infty \{1-F(p, q, \lambda_1 t, \lambda_2 t; w)\} dw}{1-F(p, q, \lambda_1 t, \lambda_2 t; 1)} = t\Phi(\lambda_1 t, \lambda_2 t). \tag{3.1}$$

Define

$$\begin{aligned} \Omega^{(p,q)}_{(\lambda_1,\lambda_2)} &= \left\{(\mu_1, \mu_2) : \boldsymbol{\lambda} \overset{rm}{\succ} \boldsymbol{\mu}, 0 < \lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2\right\} \\ &= \left\{(\mu_1, \mu_2) : \frac{p}{\mu_1} + \frac{q}{\mu_2} \leq \frac{p}{\lambda_1} + \frac{q}{\lambda_2}, 0 < \lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2\right\}. \end{aligned}$$

To prove the theorem, we need to show that for any $(\mu_1, \mu_2) \in \Omega^{(p,q)}_{(\lambda_1,\lambda_2)}$, $\varphi_{\boldsymbol\lambda}(t) \geq \varphi_{\boldsymbol\mu}(t)$ for any $t > 0$, which, by (3.1), is equivalent to $\Phi(\lambda_1 t, \lambda_2 t) \geq \Phi(\mu_1 t, \mu_2 t)$ for any $t > 0$, or $\Phi(\lambda_1, \lambda_2) \geq \Phi(\mu_1, \mu_2)$ by redefining $\lambda_1, \lambda_2, \mu_1$, and μ_2.

We claim that to prove the theorem, it suffices to show that function $\Phi(x_1, x_2)$ is decreasing along the vector fields $(1, 0)$ and $(qx_1^2, -px_2^2)$. In fact, for any $(\mu_1, \mu_2) \in \Omega^{(p,q)}_{(\lambda_1,\lambda_2)}$, if $p/\mu_1 + q/\mu_2 = p/\lambda_1 + q/\lambda_2$, then, (μ_1, μ_2) is on the curve $p/x_1 + q/x_2 = p/\lambda_1 + q/\lambda_2$, which is the integral curve of the vector fields $(qx_1^2, -px_2^2)$. If the function $\Phi(x_1, x_2)$ is decreasing along the vector fields $(qx_1^2, -px_2^2)$, we have $\Phi(\lambda_1, \lambda_2) \geq \Phi(\mu_1, \mu_2)$. If (μ_1, μ_2) is inside the region, we can find a point (μ_1', μ_2) on the curve $p/x_1 + q/x_2 = p/\lambda_1 + q/\lambda_2$. Note that we must have $\mu_1' < \mu_1$ by (1.2). Then, since the $\Phi(x_1, x_2)$ is decreasing along with the vector fields $(qx_1^2, -px_2^2)$ and $(1, 0)$, we have $\Phi(\lambda_1, \lambda_2) \geq \Phi(\mu_1', \mu_2) \geq \Phi(\mu_1, \mu_2)$ as desired.

We proceed to demonstrate that function $\Phi(x_1, x_2)$ is decreasing along the vector fields $(1, 0)$ and $(qx_1^2, -px_2^2)$. We use $A \overset{\text{sgn}}{=} B$ to denote that A and B are of the same sign.

Calculate

$$\frac{\partial F(p, q, x_1, x_2; t)}{\partial x_1} = \frac{p}{x_1}\eta(x_1 t)F(p, q, x_1, x_2; t),$$
$$\frac{\partial F(p, q, x_1, x_2; t)}{\partial x_2} = \frac{q}{x_2}\eta(x_2 t)F(p, q, x_1, x_2; t),$$

where $\eta(t) = \frac{te^{-t}}{1-e^{-t}}$. From now on, we suppress $F(p, q, x_1, x_2; t)$ as $F(t)$ for simplicity.

The gradient of $\Phi(x_1, x_2)$ along vector $(1, 0)$ is

$$\begin{aligned}\nabla\Phi_{(1,0)} &\overset{\text{sgn}}{=} \frac{p}{x_1}\int_1^\infty \{F(1)\eta(x_1)(1 - F(t)) - (1 - F(1))\eta(x_1 t)F(t)\}\,dt \\ &= \frac{p}{x_1^2}\int_{x_1}^\infty \{F(1)\eta(x_1)\,(1 - F\,(u/x_1)) - (1 - F(1))\eta(u)F\,(u/x_1)\}\,du \\ &= \frac{p}{x_1^2}\int_{x_1}^\infty K_1(u)du.\end{aligned}$$

For $u \geq x_1$,

$$\begin{aligned}K_1(u) &= F(1)\eta(x_1)\,(1 - F\,(u/x_1)) - (1 - F(1))\eta(u)F\,(u/x_1) \\ &\overset{\text{sgn}}{=} \frac{1 - F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1 - F(1)}{F(1)\eta(x_1)} \\ &= H(p, q, x_2/x_1; u) - H(p, q, x_2/x_1; x_1) \leq 0,\end{aligned}$$

by Lemma 2.1. Thus, $\nabla\Phi_{(1,0)} \leq 0$.

Define

$$K_2(u) = F(1)\eta(x_2)\,(1 - F\,(u/x_2)) - (1 - F(1))\eta(u)F\,(u/x_2)\,.$$

Then, the gradient of $\Phi(x_1, x_2)$ along vector $(qx_1^2, -px_2^2)$ is

$$\begin{aligned}\nabla\Phi_{(qx_1^2, -px_2^2)} &\overset{\text{sgn}}{=} pq\left\{\int_{x_1}^\infty K_1(u)du - \int_{x_2}^\infty K_2(u)du\right\} \\ &\overset{\text{sgn}}{=} \int_{x_1}^{x_2} K_1(u)du + \int_{x_2}^\infty [K_1(u) - K_2(u)]du \\ &\leq \int_{x_2}^\infty [K_1(u) - K_2(u)]du \\ &\leq \int_{u\geq x_2, K_2(u)\leq 0} [K_1(u) - K_2(u)]du.\end{aligned}$$

Assume that $K_2(u) \leq 0$. Since $F(u)$ is positive and increasing, we have

$$\begin{aligned}K_2(u) - K_1(u) &= \left\{\frac{1 - F(u/x_2)}{F(u/x_2)\eta(u)} - \frac{1 - F(1)}{F(1)\eta(x_2)}\right\}F(1)\eta(x_2)F(u/x_2)\eta(u) \\ &\quad - \left\{\frac{1 - F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1 - F(1)}{F(1)\eta(x_1)}\right\}F(1)\eta(x_1)F(u/x_1)\eta(u) \\ &\geq \left\{\frac{1 - F(u/x_2)}{F(u/x_2)\eta(u)} - \frac{1 - F(1)}{F(1)\eta(x_2)}\right\}F(1)\eta(x_2)F(u/x_1)\eta(u) \\ &\quad - \left\{\frac{1 - F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1 - F(1)}{F(1)\eta(x_1)}\right\}F(1)\eta(x_1)F(u/x_1)\eta(u)\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{sgn}}{=} \left\{ \frac{1-F(u/x_2)}{F(u/x_2)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_2)} \right\} \eta(x_2) \\
&\quad - \left\{ \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)} - \frac{1-F(1)}{F(1)\eta(x_1)} \right\} \eta(x_1) \\
&= \frac{1-F(u/x_2)}{F(u/x_2)\eta(u)}\eta(x_2) - \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)}\eta(x_1) \\
&= \frac{1-F(u/x_2)}{F(u/x_2)\eta(x_1u/x_2)} \frac{\eta(x_1u/x_2)\eta(x_2)}{\eta(u)} - \frac{1-F(u/x_1)}{F(u/x_1)\eta(u)}\eta(x_1) \\
&\geq \{H(p,q,x_2/x_1;x_1u/x_2) - H(p,q,x_2/x_1;u)\}\,\eta(x_1) \geq 0,
\end{aligned}
$$

where the second to last inequality is because function $\eta(x_1u/x_2)/\eta(u)$ is increasing over $u \geq x_2$, hence $\eta(x_1u/x_2)/\eta(u) \geq \eta(x_1)/\eta(x_2)$, or equivalently, $\eta(x_1u/x_2)\eta(x_2)/\eta(u) \geq \eta(x_1)$, and the last inequality is by Lemma 2.1. Therefore,

$$
\nabla \Phi_{(qx_1^2,-px_2^2)} \leq \int_{u\geq x_2, K_2(u)\leq 0} [K_1(u) - K_2(u)]du \leq 0,
$$

which completes the proof of the theorem. □

4. Discussion

In this paper, we provide an affirmative answer to Open Problem 2 raised by Balakrishnan and Zhao (2013). Interestingly, we do not use condition (1.2) explicitly in our proof. Instead, we prove that function $\Phi(x_1, x_2)$ decreases along $(1, 0)$ and $(qx_1^2, -px_2^2)$ for any $x_2 \geq x_1$, which turns out to be equivalent to $(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2) \overset{rm}{\succ} (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2)$ plus condition (1.2).

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