Graph-theoretic Asset Pricing*

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Abstract

Exploiting and explaining hidden structure in the risk correction process, our graph-theoretic approach offers novel tools and a new rule for asset pricing in the finite-state model under market completeness, one-period settlement and nontriviality of risk. Defined jointly on wealth space, pricing expected return, and its newly-defined offspring, spreads space, pricing risk, our *spreads mean-variance rule* (SMVR) represents an economics-based application of Strang's multidisciplinary *Framework for Applied Mathematics*. Five spread-related variables – atomistic risks, their bounded shadow prices and spread scales, signs and interactions – distinctively reinterpret existing accounts of multistate risk correction.

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1 Preliminaries

1.1 Asset pricing rules reconsidered

Through a reconsideration of asset pricing rules derived within the standard state-contingent asset pricing model – for example, Dothan (1990) or Duffie (1992) – this paper elaborates a new, graph-theoretic, approach to asset pricing. By revealing hidden structure in the risk correction process, a process which generally produces an offset to an asset's expected return, we adduce a new asset pricing rule: the *spreads mean-variance rule* SMVR. This rule makes the shadow pricing of risk transparent. It complements the standard account of risk neutral pricing in which an essentially risk-averse universe is transformed into a risk neutral look-alike, thereby concealing the shadow pricing of risk within implied shadow prices for wealth in each state.

In contrast, the spreads mean-variance rule prices expected return on wealth space but prices risk correction on *spreads space*, a newly defined space induced from wealth space and expressly manifesting the separate sources of risk bearing on asset pricing. For *spreads*, or payoff differentials for each pair of states, signifying those separate sources, asset risk corrections comprise the sum across all spreads of the quantum of risk defined by the signed scale of each spread valued at its *atomistic spread correction*. In turn, that atomistic spread correction represents *atomistic risk* for that spread valued at its *shadow price*. Apart from sign, atomistic risk for each spread is measured by the *unit state covariance* implied by the related pair of Arrow-Debreu assets.

Five spreads-based factors thus reinterpret existing accounts of multistate risk correction: atomistic risks, their shadow prices and the signed scale of and interactions amongst spreads. Three of these factors (spread scale, sign and interactions) serve only to distinguish assets. For each spread, the other pair of factors determines an atomistic spread correction representing a risk pricing invariant embedded in the pricing of all assets. A central preoccupation of this paper, this shadow pricing of atomistic risk represents, when compared with the standard rule set, a composite function of rule parameters distinctively different in its definition, nature and role. Provided risk is nontrivial – requiring that preferences be risk averse, that future aggregate wealth be different in each state and that no subjective probability have unit mass – well-behaved pricing rules, guaranteeing admissible prices that satisfy the non-arbitrage condition under arbitrary nontrivial choice of subjective measure, will require each shadow price for atomistic risk to be bounded between zero and unity.

The striking but hitherto hidden properties of the risk correction process reflected in such spreads-based pricing results rely for their expression on distinctive *zero sum operators.* Isolating those operators suggested to us some deeper underlying structure we expected must have found expression elsewhere in the scientific literature. Extensive search showed that to indeed be the case. In fields as varied as physics, engineering, biology, sociology and internet search engines, similar operators will be found. A major impetus towards unification of their role has been the accelerating development in recent decades of the theory of complex networks having a graph-theoretic basis. In that context, the so-called *combinatorial* or *graph Laplacian* subsumes the zero sum operators underwriting our derivation of the spreads mean-variance rule.

Accessing that graph-theoretic literature allows us to embed asset pricing rules in a canonical and multidisciplinary modelling framework – or *Framework for Applied Mathematics* – elaborated by Strang (1986, 2006), and to apply a richer corpus of analytical tools. Specializing *Strang's Framework* to asset pricing, we express our spreads mean-variance rule SMVR as the *graph Laplacian metamodel for risk correction.* As developed in Preston and Preston (2005b), this metamodel complements the *stochastic discounting metamodel for risk-adjusted pricing* – our labelling for the application to the finite-state model of the asset pricing framework formulated by Cochrane (2002). Spectral analysis, a distinguishing tool of the graph-theoretic approach, is used here to explore the structure of the *risk correction graph* representing our metamodel. Its binary disintegration in connectivity under increasing degeneracy in the subjective measure leads ultimately to the inference of bounded shadow prices applicable to the other factor, atomistic risks, together defining the atomistic spread corrections as pricing invariants for the multistate model.

1.2 Modelling framework

Asset pricing rules fall into one of two rule sets. As defined for our purposes, *risk-corrected rules* apply a risk correction (generally negative, but allowably positive) as an additive component of an asset's price in conjunction with an expected return component. CAPM, the *capital asset pricing model*, exemplifies such rules. *Risk-adjusted rules* directly adjust either the *subjective probability measure* $SPM = \phi$ or the asset's payoffs x, determining asset price post-adjustment as an expected present value. Risk neutral pricing typifies the risk-adjusted rule set.

Risk-adjusted rules have been center-stage in the state contingent model of asset pricing. For example, for wealth space $\mathcal{W}^n$ induced by payoffs to n states of nature, the simplest one-period version of the complete markets model formulated by Dothan (1990) determines endogenous risk neutral prices $q \in \mathcal{W}^n$ as a function of exogenous asset prices $p^* \in \mathcal{W}^n$ via the equilibrium pricing system $D^T q = p^*$. The dual payoffs operator $D : \mathcal{W}^n \rightarrow \mathcal{W}^n$ specifies the wealth transformation system $D\theta = W$, for terminal wealth $W \in \mathcal{W}^n$ and (assuming spanning assets equal in number to n) for $\theta \in \mathcal{W}^n$ designating portfolio positions in those assets satisfying the portfolio budget constraint $p^* \cdot \theta = W_0$. Here the scalar W_0 represents known initial-period wealth and dot notation designates the inner product of vectors. Absent arbitrage, the implied *risk neutral measure* $\pi = Rq \in \mathcal{W}^n$ provides the *risk neutral pricing* (RNP) rule for pricing any asset or set of payoffs $x \in \mathcal{W}^n$ as

$$\text{\textit{RNP: }} Rp\left(x\right) = \pi \cdot x, \tag{1}$$

for scalar asset price $p\left(x\right)$, scalar discount factor $R = 1 + r$ and known one-period risk-free rate of return r.

Reflecting the distinction by Cochrane (2002) between *absolute* versus *relative pricing* applications of asset pricing rules, Dothan's model determines risk neutral prices under a relative pricing approach. Here we provide an absolute pricing determination in terms of spreads-based pricing. Because the spreads mean-variance rule belongs to the risk-corrected rule set, we start from CAPM as the exemplar for risk-corrected rules. CAPM will be taken in the format of a *mean-variance rule* (MVR). Let W be any arbitrary *market portfolio* on the aggregate terminal wealth frontier with scalar price $p(W) = W_0$. Let that price (effectively, current period aggregate wealth) calibrate the following mean-variance rule for choice of market portfolio, conditional on choice of subjective probability measure $SPM = \phi \in \mathcal{W}^n$, as:

$$\textit{MVR-W: } Rp(W) = E_\phi[W] - \lambda VAR_\phi[W]. \tag{2}$$

Designate the positive scalar λ as the *unit risk premium* residually constraining such choices as

$$\lambda \equiv \frac{E_\phi[W] - Rp(W)}{VAR_\phi[W]} > 0, \tag{3}$$

where $VAR_\phi[W]$ defines the undiversifiable risk inhering in the market portfolio W. The positivity condition, synonymous with the presence of risk aversion, stipulates that risk-averse investors will hold the risky market portfolio only if compensated for doing so by a positive risk premium: $E_\phi[W] - Rp(W) > 0$. For CAPM, this parameter represents the first of three elements determining atomistic spread corrections.

Under linearity of pricing, the mean-variance rule

$$\textit{MVR-x:} \quad Rp(x) = E_\phi[x] - \lambda COV_\phi[x, W] \tag{4}$$

will price an arbitrary asset or portfolio x against the market portfolio W. Expressed in functional form as $p = MVR[\phi, \lambda, W;\ x, R]$, mean-variance pricing of x entails a single equation in half a dozen 'parameters'. Isomorphic pricing propositions thus depend unavoidably on how the indeterminacy thus implied is to be resolved under an absolute rather than relative pricing approach. Expressed differently, the spreads mean-variance rule to be derived from (4) must inherit the properties of the base rule specification for CAPM and hence be infected by its unresolved indeterminacy. Resolution of that indeterminacy under an appropriate existence criterion leads to the pivotal role of bounded shadow pricing of atomistic risk.

Throughout, the context entails a one-period (or this period/next period) model; asset markets are assumed both complete and fully informed; investors are assumed to possess homogeneous subjective expectations; and there is a finite number of states of nature. Touched on earlier, three mutually exclusive global risk filters – comprising the presence of maximal degeneracy in the subjective measure, the absence of uncertainty in the pricing benchmark and the absence of risk aversion in utility functions – will render risk correction and rule isomorphism trivial. Where those filters are not excluded under conditions for the nontriviality of risk, we demonstrate that either atomistic risks or their shadow prices must vanish identically across all spreads.

On notational matters, we mostly use dot notation for inner products and refer to the finite-state model as $\mathcal{S}_n$. We do not distinguish notationally between scalar random variables and their vector representations. For example, the expectation $E_\phi[x]$ is synonymous with the inner product $\phi \cdot x$. In the simple one-period context, we avoid frequent and cumbersome denominators arising from discounting by writing pricing rules in forward-value terms: references to $Rp(x)$ or its components as being forward values will recall this distinction. Proofs of theorems, where required, are collected in Appendix A.

1.3 Overview

Seven sections follow. Section 2 clarifies the hidden structure of the risk correction process, defining the induction of spreads space $\mathcal{D}^{m(n)}$ from wealth space $\mathcal{W}^n$, isolating its distinctive zero sum operators in the mappings between those spaces, and deriving the spreads mean-variance rule SMVR for $\mathcal{S}_n$. Section 3 locates the zero sum operators instrumental to the derivation of SMVR within a graph-theoretic specification of Strang's *Framework for Applied Mathematics*, formulating our *graph Laplacian metamodel for asset risk correction* as the specialization of that framework to asset pricing.

Section 4 harvests some key implications of spreads-based pricing of risk in the multistate model, including its five-factor explanation for risk correction. That section emphasizes a reinterpretation of CAPM and its isomorphs as being required to explain the shadow pricing of atomistic risks, which jointly define spread invariants relevant to the pricing of risk for all assets.

Section 5 specifies the *risk correction graph* matching the graph Laplacian metamodel and captures the binary disintegration of its connectivity under increasing degeneracy of $SPM = \phi$. Section 6 introduces, as analogue to spreads space, the risk correction partitioning of wealth space generated by the risk pricing operator – or the differential of the subjective and risk neutral measures.

Section 7 demonstrates how the spread corrections embedded in $\mathcal{D}^{m(n)}$ are reflected in specific properties for the risk pricing operator defined on wealth space, implying bounded shadow pricing of atomistic risks in the multistate model. Section 8 recapitulates key aspects of these successive steps and acknowledges some unfinished business.

2 Hidden structure of risk correction

Section 2 reveals the hitherto hidden structure of the risk correction process, now reinterpreted as transforming payoffs defined on wealth space into spreads defined on spreads space. Successive transformations of the risk correction term of CAPM utilize distinctive zero sum operators to generate a set of risk correction identities from which the isomorphic spreads mean-variance rule will be derived.

2.1 Spreads space

Asset payoffs and expected return functionals are defined on the n-dimensional wealth space $\mathcal{W}^n$ determined by the number of states. In risk-corrected rules such as CAPM, risk corrections can be viewed as being determined on the m-dimensional *spreads space* $\mathcal{D}^{m(n)}$, a space induced from wealth space that directly manifests the intrinsic sources of risk in the state-contingent model. The n payoffs of the market portfolio $W > 0$ generate $m(n) \equiv n(n-1)/2$ *spreads* $\triangle W_{ji}$ between each distinct pair of payoffs. For $n > 3$, $m(n) > n$ – so that the dimension of spreads space typically exceeds the dimension of wealth space. The cases $n = \{2, 3\}$, for which $m(n) = \{1, 3\}$, are therefore dimensionally atypical.

Since spreads are unique up to their sequence and sign, impose on the market portfolio the *natural ordering* $W_j > W_i$, $j > i$, progressively ordering payoffs (and hence labelling states) from low wealth outcomes to high wealth outcomes. Assisting in signing shadow prices for atomistic risk, under this convention CAPM market spreads can be defined to be universally positive without loss of generality:

$$\textit{Natural ordering convention: } \Delta W_{ji} \equiv W_j - W_i > 0 \quad (j > i;\ j \textit{ iterated first}). \tag{5}$$

Express any market portfolio as:

$$W = W_1 s + W'_\Delta; \qquad W'_\Delta \equiv \left(\Delta W_{i(i-1)} \mid \Delta W_{10} \equiv 0\right), \quad (i = 1, \ldots, n). \tag{6}$$

Here s represents the *unit bond*, or safe asset, returning a dollar certain and spanning the *safe asset cone*

$$\textit{Safe asset cone: } SAC^n \equiv Sp\{s\}. \tag{7}$$

Of the n degrees of freedom in the specification of W (subject to natural ordering), one goes to specify the bond position $W_1 s$ or the base scale of W and $n-1$ determine the $n-1$ *natural spreads*

$$W_\Delta \equiv \left\{\Delta W_{(i+1)i} > 0 \mid i = 1, \ldots, n-1\right\} \tag{8}$$

representing the successive first differences of the market payoffs. Accordingly, although there are $m(n) = n(n-1)/2$ spreads, only $n-1$ of them can be specified independently.

The origin of spreads space represents all assets or portfolios satisfying the condition $\triangle x = 0$: in other words, all *safe assets* defined as exhibiting no variation in their payoffs across states. Any other admissible point in spreads space necessarily represents a unique set of risky assets having the same configuration of nonvanishing asset spreads $\triangle x \neq 0$. Not all points in spreads space are reachable – consider the universally inadmissible spreads $\Delta x = \mathbf{1}$ for $\mathbf{1}$ denoting a unit vector of appropriate dimension. In general, references to spreads $\Delta x \in \mathcal{D}^{m(n)}$ must refer to admissible rather than arbitrary spreads. Because all safe assets S possess vanishing spreads, all risky assets $x + S$ represent risks equivalent to x as having the same spreads $\triangle x \neq 0$. The source of risk is payoff variability between states: when there are n payoffs there are necessarily $m = m(n)$ such sources. Each source potentially attracts a different price for its risk. Understanding those differences is our ultimate goal.

2.2 Zero sum operators

CAPM (4) is specified solely in wealth space, in terms of asset payoffs, x and W. To examine risk correction in spreads space, we progressively transform the covariance component $COV_\phi[x, W]$ into a bilinear form in the vector spreads, $\triangle x$ and $\triangle W$, of this asset pair. First express that covariance as the following bilinear form in asset payoffs:

$$COV_\phi[x, W] = E_\phi[xW] - E_\phi[x]E_\phi[W] = W^T\Phi x - \left(W^T\phi\right)\left(\phi^T x\right) = W^T\Psi x, \quad (9)$$

for *subjective probability operator* $\Phi \equiv diag[\phi]$ and for symmetric $n \times n$ *covariance payoffs operator* $\Psi \equiv \Phi - \phi\phi^T$ specified in Appendix A.1. Two features completely define Ψ for any choice of state dimension n. Firstly, its diagonal elements for each state are the respective products of the subjective probability ϕ_s for that state and its probability complement $1-\phi_s$; and, second, its symmetric off-diagonal elements are the negative products of the subjective probabilities for the relevant pair of states.

After substituting for the probability complements $\overline{\phi_s} \equiv 1 - \phi_s = \sum_{j\neq s}\phi_j$, observe that the rows and columns of the symmetric operator $\Psi = \Psi^T$ sum to zero. Where rows or columns (or, as here, both rows and columns) demonstrate this property, such matrices will be described as *zero sum operators*. Satisfactorily explaining the role of zero sum operators will lead us to the graph-theoretic approach set out in Section 3.

To be highlighted from the outset, a zero *row sum* operator has two important effects. Firstly, it annihilates constant vectors: for example, all safe assets S belong to the nullspace of the covariance payoffs operator Ψ and so attract a zero risk correction. Second, it transforms all other vectors from levels to differences: hence, under the covariance operator Ψ, all risky assets have their payoffs transformed into spreads. Similarly, in the process of risk correction, zero *column sum* operators impose a consistency requirement ensuring that risk corrections are distributed across spreads with zero sum. Zero row sum operators are central to this section; zero column sum operators will be central to Section 7.

Having variable mixed-sign patterns of variably-scaled spreads, arbitrary assets or portfolios will not generally satisfy the natural ordering convention (5). As a second application of zero row sum operators, parallel the natural ordering convention with the following *spreads convention* in order to ensure that all assets have their spreads measured consistently (see Appendix A.2 for further illustration).

Definition 1 Spreads convention: *Under the spreads convention, asset payoffs $x \in \mathcal{W}^n$ and their induced spreads $\Delta x \in \mathcal{D}^{m(n)}$ are related by the differencing operator $x \xrightarrow{B} \Delta x$ mapping from wealth space into spreads space, so that $\Delta x \equiv Bx$ where B is an $m \times n$ zero row sum matrix whose only nonzero elements in each row are ∓ 1 in the i'th and j'th columns corresponding to the spread definition $\triangle x_{ji} \equiv x_j - x_i$ $(j > i;\ j$ iterated first).*

Observe that (5) complies with the spreads convention but also enforces natural ordering $0 < W_1 < \ \cdots < W_n$ to guarantee that all market spreads thus generated, including the natural spreads (8), are also positive. Importantly, the ambiguities in either convention concerning the sign and sequencing of spreads correspond to immaterial permutations, later shown to wash out in the formulation of the *Strang Framework* (see Section 3.2). Reflecting the available degrees of freedom in specifying spreads, the $m \times n$ differencing operator B induced by this spreads convention must have rank $n - 1$. This zero row sum differencing operator now proves instrumental in distentangling the progressive transformations of the risk correction covariance which lead to the spreads mean-variance rule.

2.3 SMVR: Spreads mean-variance rule

Variously expressing either weighted or unweighted asset covariances in terms of payoffs and/or spreads, the following theorem (see Appendix A.3 for proof) demonstrates the flexibility of zero sum operators and, in doing so, begs an explanation for that flexibility.

Theorem 2 Risk correction isomorphisms: *For n payoffs and $m \equiv n(n-1)/2$ spreads, the risk correction applied to asset x under pricing portfolio W can be expressed isomorphically in terms of payoffs, mixed payoffs and spreads, or spreads, as follows:*

$$\lambda COV_{\phi}[x, W] = \lambda \left(W - \overline{W}\right)^T \Phi x = \lambda W^T \Psi x = \lambda \Delta W^T \Sigma^T x = \lambda \Delta W^T \Pi \triangle x, \tag{10}$$

where:

(i) $\overline{W} \equiv E_{\phi}[W]$ and the $n \times n$ operator $\Phi \equiv diag[\phi]$ defines the subjective probability operator;

(ii) Ψ is the symmetric $n \times n$ zero row and column sum covariance payoffs operator with the specific diagonal and off-diagonal definitions recorded in Appendix A.1 and defined by

$$\Psi \equiv \Phi - \phi\phi^T = B^T \Pi B; \tag{11}$$

(iii) the $m \times n$ differencing operator $x \xrightarrow{B} \Delta x$ specified under the spreads convention (Definition 1) represents a zero row sum mapping from wealth space $\mathcal{W}^n$ into spreads space $\mathcal{D}^{m(n)}$;

(iv) the weighted differencing operator $\Sigma^T = \Pi B$ is an $m \times n$ zero row sum operator defined by the property that the row of Σ^T matching the spread $\triangle W_{ji} \in \triangle W$ has only two nonzero entries $\left(-\phi_i\phi_j,\ \phi_j\phi_i\right)$, appearing in the $i'th$ and $j'th$ columns respectively; and

(v) the state covariance operator Π is an $m \times m$ diagonal matrix with diagonal elements $\phi_i\phi_j$ corresponding to each pair of matched spreads $\triangle W_{ji}$, $\triangle x_{ji}$.

The spreads-based definition for the weighted covariance chained at the end of (10) enables the pricing of risk in the multistate model to be intuitively re-expressed. The bilinear form $\triangle W^T \Pi \triangle x$ expresses the covariance between any pair of assets (somewhat loosely put) as the sum of their m 'spread covariances'. Weighted by the unit risk premium λ, those spread covariances will determine the sign and magnitude of risk correction. Our new rule – the spreads mean-variance rule – follows directly from substituting the final expression of (10) into the mean-variance rule (4).

Theorem 3 *Spreads mean-variance rule: Defined jointly on wealth space $\mathcal{W}^n$ (pricing expected return) and spreads space $\mathcal{D}^{m(n)}$ (pricing risk), the spreads mean-variance rule is the expected present valuation*

$$SMVR:\quad Rp\left(x\right) = \phi \cdot x - \mu \cdot \triangle x \tag{12}$$

having positive (nonnegative) atomistic spread corrections $\mu \in \mathcal{D}^{m(n)}$ defined by

$$\text{Atomistic spread corrections:}\quad \mu \equiv \lambda \Pi \triangle W = \lambda \Sigma^T W, \tag{13}$$

for nontrivial risk arising under the strongly (weakly) nondegenerate subjective probability measure $SPM = \phi > 0$ ($\phi_s \neq 1$), risk averse preferences $\lambda > 0$ and stochastic market portfolio W, $\triangle W > 0$.

Under spreads mean-variance pricing, the price of any asset or portfolio resolves into the difference of a pair of inner products, the first being the standard expected total return in terms of the asset's payoffs x evaluated under the subjective measure $SPM = \phi$. The second inner product defines asset risk correction in terms of asset spreads $\triangle x$ evaluated under the positive ($\phi > 0$) or nonnegative ($\phi_i \neq 1$) *atomistic spread corrections* $\mu \in \mathcal{D}^{m(n)}$ applying to each of the m spreads. Their designation stems from the following definition of *atomistic risks* for $\mathcal{S}_n$ (see Appendix A.4 for proof of (14)).

Definition 4 *Atomistic risks: Under the spreads mean-variance rule, atomistic risks may be specified in spreads space $\mathcal{D}^{m(n)}$ as the absolute values of the m distinct unit state covariances, for those covariances defined in respect of each spread as:*

$$\text{Unit state covariances:} \qquad usc_{ji} \equiv COV_\phi [\delta_j, \delta_i] = \Delta\delta_j^T \Pi \Delta\delta_i = -\phi_j \phi_i, \quad \begin{cases} j = 2, \ldots, n \\ i = 1, \ldots, n-1 \\ j > i; \ j \text{ iterated first,} \end{cases} \tag{14}$$

indexed here on wealth space $\mathcal{W}^n$, and with $\{\delta_s \mid s = 1, ..., n\}$ the set of Arrow-Debreu assets returning mutually exclusive dollars in the relevant state and nothing otherwise. Alternatively indexed on spreads space $\mathcal{D}^{m(n)}$, these m atomistic risks comprise the diagonal elements of the state covariance operator Π, so that

$$\alpha(k) \equiv \Pi_{kk} \equiv |usc(k)|, \qquad k = 1, \ldots, m(n). \tag{15}$$

Focusing on the deeper structure now evident in the risk correction process, Section 3 will also motivate this definition ahead of a first exploration of spreads-based pricing in Section 4.

3 Graph-theoretic pricing

Transforming asset covariances (10) from a bilinear form in asset payoffs to a bilinear form in asset spreads sees the symmetric zero sum operator $\Phi - \phi\phi^T$ of the first form become the triple matrix product $B^T \Pi B$, necessarily symmetric and zero sum, implied by the second form: so that

$$COV\phi[x, W] = W^T \Psi x = W^T \left[\Phi - \phi\phi^T\right] x = (\Delta W)^T \Pi \Delta x = W^T \left[B^T \Pi B\right] x, \tag{16}$$

for zero row sum operator B determined by the spreads convention. Nominally confined to wealth space, the mapping $\Phi - \phi\phi^T$ conceals successive mappings $B^T \Pi B$ from wealth space to spreads space, on spreads space, and from spreads space back to wealth space. The zero sum symmetry of the covariance operator Ψ must therefore signal concealed structure in the risk correction process, which the spreads mean-variance rule SMVR of (12) partially reveals by unmasking the role of spreads space. But it does not lay that concealed structure completely bare since SMVR expresses the weighted bilinear spreads-based form $(\lambda\Delta W)^T \Pi\Delta x$ as the risk correction functional $\mu \cdot \Delta x \equiv (\lambda\Pi\Delta W) \cdot \Delta x$ defined on spreads space.

Strang's *Framework for Applied Mathematics* – see Strang (1986, 2006) – systematically assimilates, in a unifying multidisciplinary modelling framework, a trio of such triple matrix products of generic structure $A^T CA$ respectively associated with the formulation, solution and dynamics of symmetric linear systems (compare for example Strang (1986), pp 2-3). Within a graph-theoretic rendering of *Strang's Framework*, this section fully resolves the role of the tripartite mappings in the covariance payoffs operator $\Psi = B^T\Pi B$.

Our specialization of that framework to asset pricing we designate the *graph Laplacian metamodel for asset risk correction*. The *graph* descriptor signals its graph-theoretic structure and with the *Laplacian* descriptor (see below) connotes its affiliation with *Strang's Framework*. The *metamodel* descriptor signifies a status in relation to the risk-corrected rule set comparable to that exhibited in relation to the risk-adjusted rule set by the *stochastic discount factor/generalized method of moments framework* formulated by Cochrane (2002). (See Preston and Preston (2005b) for discussion of asset pricing metamodels in terms of these two frameworks.)

3.1 Strang's *Framework for Applied Mathematics*

Strang (1986, 2006) defines a canonical framework for applied mathematical analysis that unifies otherwise disparate modelling across a myriad of fields and also enables application of a richer corpus of analytical tools. When given a graph-theoretic representation, *Strang's Framework* associates vectors of variables relevant to some field of study to the vertices, edges and weighted edges of a related graph. Subject to some necessary variation in detail, a generic trio of equations relates vertices to edges, edges to weighted edges, and weighted edges to equilibrium outcomes via some external forcing function.

Let v denote a set of vertices, e of edges, e^* of weighted edges and f a vector forcing function that exogenously determines the relevant behaviour of some open system of interest. Then *Strang's Framework* entails the following three equations.

$$\text{Strang's } \textit{Framework for Applied Mathematics}: \quad \left\{ \begin{array}{rcl} e & = & Bv \\ e^* & = & Ce \\ B^T e^* & = & f. \end{array} \right. \quad \Leftrightarrow \quad \left. \begin{array}{c} Lv = f \\ \left(L \equiv B^T C B\right) \end{array} \right\} \tag{17}$$

Strang's examples illustrate how boundary conditions, when relevant, may affect the coefficients of the operators appearing in (17). Strang also emphasizes that particular applications may require some modification to this framework, so that it offers a unifying starting point without being conclusive.

Under backward substitution from the third equation for e^* and then e, *Strang's Framework* generates the so-called *Laplacian system* $Lv = f$ mapping vertices v to the exogenous forcing function f. Strang refers to $e^* = Ce$ as the *constitutive law* for the system, and to C as the *constitutive matrix*. This law captures how edges must be differentially weighted to reflect relevant physical or other modelling considerations. He refers to $B^T e^* = f$ as an *equilibrium* or *balance condition*. In essence, it captures how an open system is represented as being embedded in its larger environment. The vertex-edge *incidence law* specifies how relevant endogenous variables represented at the vertices are transformed into further endogenous variables associated with the edges, having relevant physical or other characteristics subject to or expressible by weighting under the constitutive law.

This Laplacian system finds widespread use in physics, engineering, biology, sociology and other areas – some recent and eclectic examples recovered from online searching include Jernigan, Demirel, and Bahar (1999), Hao, Xie, and Zhang (2001), Cremean and Murray (2003) and Grady and Schwartz (2003). This is not to say that all such uses expressly recognize their embedding in *Strang's Framework*; nevertheless, they all represent implicit or explicit examples of that framework. Strang (1986) provides many further examples, including least squares regression. His forthcoming book, Strang (2006), further develops that framework. We stress that our asset pricing application expressly taps only one of the framework's trio of triple matrix products.

This very summary introduction raises some obvious questions. Firstly, what are the essential graph-theoretic concepts supporting the Laplacian system $Lv = f$? Second, how does the Laplacian system help motivate the roles of spreads space, its related zero sum operators and their ultimate expression in the triple matrix product $B^T \Pi B$? We then demonstrate how the risk correction process reflected in the spreads mean-variance rule SMVR can be fitted comfortably into *Strang's Framework*.

3.2 Graph-theoretic approach

Our exposition of graph theory here relies particularly on the two surveys of graph Laplacians by Merris (1994, 1995) – compare, also, Strang (1986), p 88. Consider the graph $\mathcal{G}(V, E)$ of $n = 4$ vertices V and $m = 6$ edges E shown in the left hand panel of Figure 1 (labelled consistently with the spreads convention). Each of the $n = 4$ vertices v_i has $n - 1 = 3$ edges incident on it. For example, the vertex v_1 anchors the three edges e_{21}, e_{31} and e_{41}. Each vertex is described as having *degree* 3, and the graph is described as being *regular*. The graph is *complete* in the sense that all possible edges between the vertices are filled in. It is *undirected*: the edge relationship defines a two-way relationship between its vertices. In the asset pricing framework, the risk generated by state uncertainty runs in both directions between any pair of states. In other applications, the direction of flow may be fixed — in which case the graph is *directed* (a *digraph*).

Now define the $n \times n$ *adjacency matrix* A, illustrated in Table 1 for the graph for $\mathcal{S}_4$, by the conditions $a_{ii} = 0$ and $a_{ij} = 1$ $(i \neq j)$. Each row relates to a vertex and counts, with a unit weight for each, the number of edges incident on that vertex. The zero weights $a_{ii} = 0$ indicate that there are no self-referencing edges in the form of loops on any vertex. The unit weights on each edge indicate that there are no parallel edges joining any pair of vertices. Each column of the adjacency matrix identifies the pattern of connections amongst the vertices: in the example, each vertex is connected to the other three vertices, so each

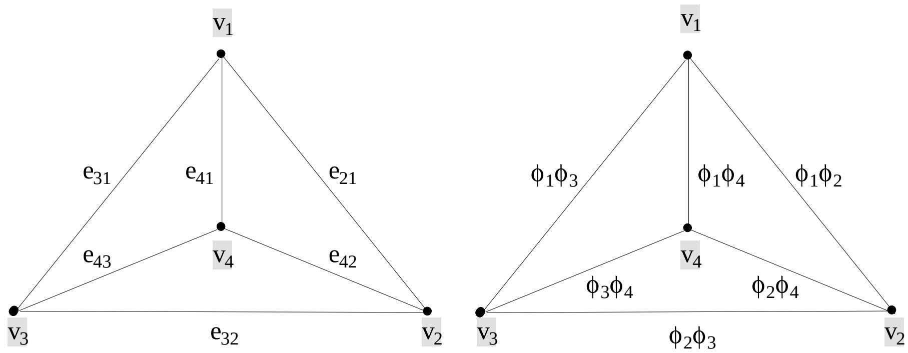

Figure 1: *Complete graph for $\mathcal{S}_4$ with $n = 4$ vertices and $m = 6$ edges either unit- or probability-weighted*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}; \qquad L = D - A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Table 1: *Adjacency and Laplacian matrices for S_4*

column also has three unit entries.

For *degree matrix* $D \equiv diag\,[d_i]$ (in the regular and complete graph of the current example, $d_i = d = 3$), define the *Laplacian matrix* L as the n-square matrix representing the difference between the degree matrix and the adjacency matrix, so that $L \equiv D - A$. For the example of Figure 1, $d = 3$ implies the matrix illustrated on the right of Table 1. Here, under this graph-theoretic recipe for construction, is a generic example of a zero row sum, zero column sum operator. In the physics of electrical circuits or of load-bearing springs, for example, the Laplacian matrix is also referred to as a *Kirchhoff matrix* after its 19th century discoverer.

As separate motivation for the definition of the Laplacian, reinterpret the components of v (at the vertices of Figure 1) as payoffs and the components of e (along the edges) as spreads. Recall the differencing operator $Bx = \triangle x$ relating vertices to edges where B is the $m \times n$ zero row sum operator induced by the spreads convention of Definition 1. For the current $\mathcal{S}_4$ example, the four vertices of the figure can be related to its six edges under the zero row sum mapping given at (39) in the illustration of that definition at Appendix A.2.

Echoing our earlier comments, this mapping is not uniquely defined, depending as it does on the precise choices about which direction along an edge is given a positive unit weight and the sequencing of edges. But these factors are permutations only, and wash out because of the countervailing effect of the adjoint mapping. Specifically, the Laplacian may also be defined by the composition of any such choice of differencing operator with its adjoint, so that

$$L = D - A = B^T B. \tag{18}$$

For example, using (39) to evaluate $B^T B$ for $\mathcal{S}_4$ verifies $L = D - A$ in Table 1. Because the differencing operator is uniquely specified up to sign and sequencing of spreads, any alternative definition B^* will be related to B via $B^* = PB$ for P a permutation operator. Hence $B^{*T} B^* = B^T P^T P B = B^T B$, since the transpose and inverse of a permutation operator are identical – see Strang (1976), p 120. The sequel will help motivate this twofold definition for the Laplacian.

3.3 Graph Laplacian for risk correction

While the Laplacian (18) illustrated by Table 1 has the properties of symmetry combined with zero row and column sums, it differs from the payoffs covariance operator Ψ (compare Appendix A.1) in having identical off-diagonal elements (-1) and identical diagonal elements $(n-1)$. Essentially, that specialized structure relative to our asset pricing application derives from the unit weights assigned to each edge of the graph.

In the graph-theoretic framework, the *generalized* or *weighted Laplacian* derives from weighting of the edges of $\mathcal{G}(V, E)$. Captured in the *constitutive matrix* C defined in *Strang's Framework* (17), those edge weightings reframe the Laplacian as

$$\textit{Generalized Laplacian:}\;\; L^* = B^T C B.$$

Illustrating such edge weightings, the second panel of Figure 1 places the risk correction process for the multistate model squarely in these graph-theoretic terms. Notice there is

$$(\mathcal{S}_3)\quad \begin{bmatrix} \phi_1(1-\phi_1) & -\phi_1\phi_2 & -\phi_1\phi_3 \\ -\phi_1\phi_2 & \phi_2(1-\phi_2) & -\phi_2\phi_3 \\ -\phi_1\phi_3 & -\phi_2\phi_3 & \phi_3(1-\phi_3) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1\phi_2 & 0 & 0 \\ 0 & \phi_1\phi_3 & 0 \\ 0 & 0 & \phi_2\phi_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Table 2: *Laplacian construction for three-state model*

nothing special about the geometric arrangement of vertices and edges or the size of n. Only the interactions between pairs of vertices connected by edges, however represented and however many, are relevant.

Applying the differencing mapping $By \equiv \triangle y$ to both the market portfolio W and the canonical risky asset x in the risk correction identities (10) transforms the covariance between the risky asset and the market portfolio as described by (16). In terms of *Strang's Framework* (17), the core of the risk correction process is the *graph Laplacian*:

$$\textit{Graph Laplacian:}\ \ \Psi = B^T \Pi B = \Phi - \phi\phi^T, \tag{19}$$

so called because the algebraic properties of this operator find a direct reflection in a correspondingly highly-structured graph representing the Laplacian system (17) – compare our more intensive analysis of the *risk correction graph* in Section 5.

In $\mathcal{S}_3$, this Laplacian construction is illustrated, under our convention for the spreads definition, by Table 2. To isolate the source of the two zero sum properties reflected in the regularities exhibited by the standard Laplacian – compare the $\mathcal{S}_4$ example of Table 1 – suppose that $SPM = \phi$ is uniformly distributed, so that $\phi_s = 1/n$. In that case, as illustrated for the $\mathcal{S}_3$ example of Table 2 by Table 3, but for the scalar weighting of all elements by $1/n^2$ the diagonal elements revert to $n-1$ and all off-diagonal elements revert to -1.

So the zero row sum property of the differencing operator $B : \mathcal{W}^n \to \mathcal{D}^m$, mapping from wealth space to spreads space, combines with the zero column sum property of its adjoint

$$\Psi = B^T \Pi B = \frac{1}{9}\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Table 3: *Graph Laplacian for S_3 under uniform distribution*

$B^T : \mathcal{D}^m \to \mathcal{W}^n$, mapping in the reverse direction, to produce the zero row and column sum properties of Ψ. Additional richness accrues to the structure of the covariance payoffs operator, when benchmarked against the standard Laplacian, to the extent therefore that the subjective distribution departs from a uniform distribution (equivalently, the diagonal state covariance operator Π departs from a scalar-weighted identity matrix). No longer are the diagonal and off-diagonal elements of Ψ respectively identical at $n-1$ and -1. The positivity of the diagonal elements and negativity of the off-diagonal elements are preserved but, subject to the symmetry and adding-up constraints, continuously varying patterns of elements are consistent with continuous variation of the subjective distribution away from a uniform distribution.

3.4 Graph Laplacian metamodel

We now redeem our promise to fit the spreads mean-variance rule (12) into *Strang's Framework* (17). Risk correction is determined endogenously as a bilinear form involving the asset spreads, which are priced against the exogenous market portfolio and unit risk premium appearing in that bilinear form in the guise of the market spreads weighted by the unit risk premium. Compared to the linear forms characterizing the generic formulation of *Strang's Framework*, the forcing function f will therefore be manifested differently in our graph Laplacian metamodel for risk correction.

Apply the notation x for n vertices, $\triangle x$ for $m\,(n)$ edges, B for the zero row sum differencing operator mapping from vertices to edges, B^T for its adjoint mapping in the reverse direction and Π for the diagonal $m \times m$ constitutive matrix. Envisage the risk correction process of asset pricing as the open system specified by Table 4. Figure 2 illustrates (lighter

Δx	$=$	Bx	*Risk domain law*
Δx^*	$=$	$\Pi \Delta x$	*Constitutive law for risk diffusion*
rc	$=$	$Bf^* \cdot \Delta x^* \quad (f^* = \lambda W)$	*Risk shadow pricing law*
$Rp\,(x)$	$=$	$\phi \cdot x - rc$	*Equilibrium asset pricing.*

Table 4: *Graph Laplacian Metamodel*

shading denotes variables endogenous to the risk correction process; darker shading refers to parameters exogenously associated with the CAPM pricing benchmark).

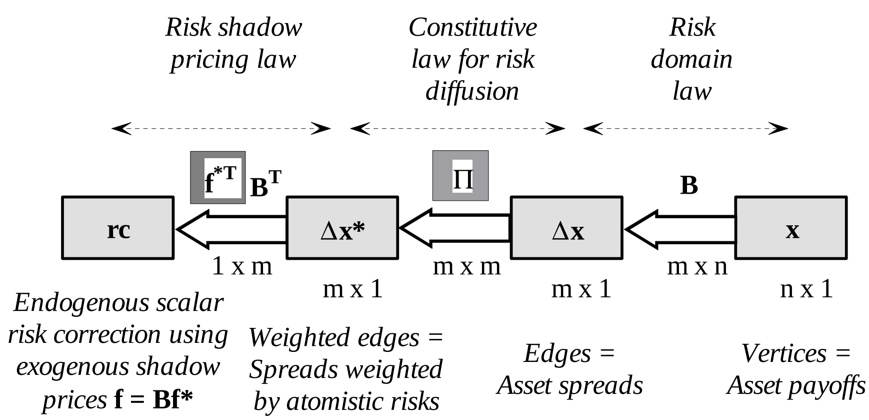

Figure 2: *Graph Laplacian metamodel for risk correction*

Risk correction is determined conceptually, not in wealth space, but in spreads space where each admissible element represents a set of equivalent risk assets. A safe asset, for which $\triangle x = 0$, attracts no risk correction whatever the magnitude of its payoffs. Only risky assets, defined by admissible $\triangle x \neq 0$, typically require risk correction (the exception being idiosyncratic risks considered later). Spreads space is accordingly the natural domain for the analysis of risk correction. The transformation $x \xrightarrow{B} \triangle x$ under the differencing operator matching the spreads convention is thus the starting point and will be referred to as the *risk domain law*.

As its naming implies, the *constitutive law* defines the constituents of the risks which are to be priced under the third law. Its m elements $\phi_i\phi_j = |COV_\phi[\delta_i, \delta_j]|, \quad i \neq j$, as specified by Definition 4 for the state covariance operator Π, are the lowest level constituent of the risk correction process. They define the atomistic risk generators as the absolute value of the covariances between mutually exclusive dollars in each of the m pairs of states, under the relevant pairwise products conditioned on the subjective measure $SPM = \phi$. Unless the subjective measure is uniformly distributed, these atomistic risks can not be uniform. Depending as they must on the subjective measure, atomistic risks inevitably have a subjective component determined by the formation of collective expectations. The second constitutive element is the *signed scale* or quantum of risk $\triangle x_k \gtreqless 0$, $k = 1, \ldots, m$, defining each of the m spreads associated with a given asset. The constitutive law for risk diffusion $\triangle x^* = \Pi \triangle x$ measures the levels of risk embedded in the given asset in terms of signed scalings of the atomistic sources of state contingent risk — namely, the product of the absolute unit state covariances with their respective spreads.

These first two laws mimic *Strang's Framework*. The third law, for equilibrium (forward) shadow pricing of risk, differs somewhat from the Strang analogue (consistent with his observation that precise details will vary with context). The difference stems from the fact that the shadow pricing relationship is bilinear rather than linear. Nevertheless, the inner product therefore associated with the shadow pricing law for risk reintroduces the adjoint operator, mapping from spreads to payoffs, analogously to Strang's third equation (compare Figure 2 with (17)).

The risk shadow pricing law differentiates the graph Laplacian metamodel from SMVR (12). For $f \equiv Bf^*$, the graph Laplacian risk correction functional is $rc = f \cdot \Delta x^*$; the SMVR functional is $rc = \mu \cdot \Delta x$. Of course, both functionals are defined on spreads space and their isomorphism is obvious. But that distinction has unfolded the tripartite structure of $\Psi = B^T \Pi B$ within Strang's *Framework for Applied Mathematics*. Compactly expressed in wealth space, the graph Laplacian metamodel becomes:

$$\textit{Compact GLM}\colon Rp\,(x) = \phi \cdot x - rc = (\phi - \Psi f^*) \cdot x. \tag{20}$$

Whether or not rendered explicit, the tripartite mappings $B^T \Pi B$ of the graph Laplacian Ψ

explain how the consolidated structure of the related asset pricing metamodel ensures rule isomorphism across the entire rule set.

4 Spreads-based pricing of risk

Section 4 now takes possession of some key implications of spreads-based pricing of risk, also formulating with their aid the agenda for ensuing sections.

4.1 Conditioning on global risk filters

Conclusions about atomistic risks and their shadow prices are conditioned on the global risk filters earlier identified (compare also (13)).

Theorem 5 Shadow pricing of atomistic risks: *Define the exogenous forcing function for the graph Laplacian metamodel as the weighted market portfolio $f^* \equiv \lambda W \in \mathcal{W}^n$ specified on wealth space. (i)* Shadow prices: *the induced specification on spreads space defines the shadow prices for atomistic risks*

$$\textit{Shadow prices: } f \equiv Bf^* \equiv \lambda \Delta W \in \mathcal{D}^{m(n)}. \tag{21}$$

(ii) Atomistic spread corrections: *applying Hadamard notation, the atomistic spread corrector appearing in (13) is then representable as $\mu \equiv f\alpha \in \mathcal{D}^{m(n)}$ where atomistic risks are defined by (15). (iii)* Vanishing spread corrections: *for all spreads, atomistic spread corrections $\mu(k) \equiv f(k)\,\alpha(k)$, $k = 1, \ldots, m(n)$, vanish identically when any one of the global risk filters $\{\lambda = 0,\ \Delta W = 0,\ \phi_i = 1\}$ applies. (iv)* Uniformly positive spread corrections: *atomistic spread corrections are uniformly positive under the strong condition $\{\lambda > 0,\ \Delta W > 0,\ \phi > 0\}$ for nontriviality of risk.*

Observe that the *weak condition* for nontrivial risk $\{\lambda > 0,\ \Delta W > 0,\ \phi_i \neq 1\}$, at its extreme, only requires a single atomistic risk to survive. While for that reason the strong condition $\phi > 0$ may typically be preferred, the weak condition proves important for analyzing the role of the $m(n)$ atomistic spread corrections in the multistate model.

4.2 Five-factor pricing of risk

Spreads-based pricing offers a transparent five-factor explanation for the pricing of risk in the multistate model. Express risk correction under SMVR as the m-fold sum of each *spread correction* sc for the canonical asset, so that

$$\begin{aligned} &\textit{SMVR:} \quad rc \equiv \mu \cdot \Delta x \;\; (= \Pi f \cdot \Delta x = B f^{*} \cdot \Delta x^{*}) \\ &= \sum_{k=1}^{m(n)} sc\,(k) \equiv \sum_{k=1}^{m(n)} \mu\,(k)\,\Delta x\,(k) = \sum_{k=1}^{m(n)} \alpha\,(k)\,f\,(k)\,sgn\Delta x\,(k)\,|\Delta x\,(k)|\,. \end{aligned} \tag{22}$$

For each of the m spread corrections, four of these factors comprise: atomistic risk, its shadow price, the scale of the risk and spread sign. In general, all four factors vary across spreads with the product of the first two, $\mu \equiv f\alpha$, nonnegative (positive) under the weak (strong) condition for nontrivial risk.

In consequence of the natural ordering on the market portfolio, each spread correction has its sign fixed by the spread sign. Transparently unfolding the well-known CAPM result, whenever a matched pair of spreads for the market portfolio and an asset being priced against it are negatively correlated because the asset spread is negative, the asset will be more valuable in consequence. The scale and number of negative spread correlations will determine how much more valuable.

The fifth explanator refers to those interactions amongst spread corrections. Appropriate mixed-sign configurations of the spread corrections will generate a vanishing risk correction, so that the relationship $rc = \sum_{k=1}^{m(n)} sc\,(k) = 0$ defines a category of risky asset – *idiosyncratic risks* – that attracts no risk correction (see Section 6.3).

4.3 Explaining the CAPM risk premium

Some twists and turns in explicating the shadow pricing of risk are given perspective by the distinction between *absolute* and *relative* pricing applications stressed by Cochrane (2002) – compare pp. *xiv*, 184. Of the five spreads-based factors explaining the pricing of risk in $\mathcal{S}_n$, spread sign, spread scale and spread interactions serve only to distinguish one asset from another. Hence the atomistic spread corrections defined by $\mu \equiv f\alpha \in \mathcal{D}^{m(n)}$, embedded in the CAPM pricing of every asset, isolate the invariant explanators for the pricing of risk. The definition of atomistic risks under the exogenous specification $SPM = \phi$ (Definition 4), as reflected in the constitutive matrix Π of the graph Laplacian metamodel (recall Figure 2), and of their shadow prices $f \equiv \lambda \Delta W$ from differencing of the exogenous forcing function, or weighted market portfolio, $f^* \equiv \lambda W$ (Theorem 5) thus implies a significant alternative interpretation of CAPM, via its risk-corrected isomorph SMVR, as a theory of asset pricing couched in terms of atomistic risks and their shadow pricing.

A comprehensive analysis of the shadow pricing of atomistic risk will here fully explain the determination of the CAPM market risk premium for any choice of market portfolio as pricing benchmark. (The remaining degree of freedom – choice of a specific market portfolio – is then to be regarded as being resolved under appropriate optimisation of consumption and wealth paths.) Various threads have to be unravelled to that end. Firstly, Section 5 ties properties of the risk correction graph to properties of the graph Laplacian, clarifying how individual spreads-based risk corrections are tied to spread-related measures embedded in $SPM = \phi$. Second, to explain the isomorphisms between the asset pricing rule set defined on wealth space and the graph Laplacian metamodel or SMVR defined on spreads space, Section 6 specifies the *risk correction partitioning* on wealth space $\mathcal{W}^n$ as the indirect analogue of spreads space $\mathcal{D}^{m(n)}$. Third, connecting those two threads, the risk pricing operator or generator of that partitioning is shown, in Section 7, to have a number of properties stemming from its hidden connection with spreads space, properties leading, in particular, to the inference of bounded shadow pricing of atomistic risks under well-behaved pricing rules.

5 Risk correction graph

The risk correction graph $\mathcal{G}_\phi$ generating the graph Laplacian $\Psi = B^T \Pi B$ possesses connectivity that disintegrates in binary fashion under increasing degeneracy in the subjective measure, where that measure determines the edge weighting for the graph. That structural sensitivity is captured in the Laplacian spectrum Λ_L and distinctive algebraic properties of the graph Laplacian. This structured disintegration of graph connectivity explains the embedded character of spreads-based pricing of risk.

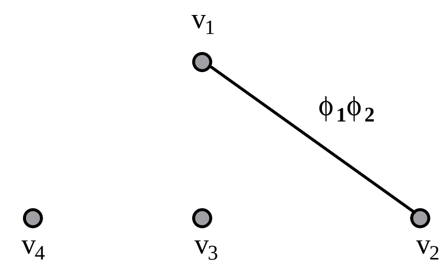

Figure 3: *Impact of measure degeneracy on graph connectivity in* S_4

Under homogeneous expectations, nondegeneracy of the subjective measure $SPM = \phi > 0$, in the *strong sense* that all subjective probabilities are positive, precludes dimensional irrelevance or inflation in the state space. To the contrary, and recalling Figure 1, allocate all probability mass to the first two states of $\mathcal{S}_4$. Figure 3 demonstrates how the measure degeneracy $\phi_3 = \phi_4 = 0$ affects graph connectivity. Five of the six edge weights have been set to zero, creating three separate *components* or disconnected sub-graphs. The risk correction graph for $\mathcal{S}_4$, conditioned on this maximally degenerate specification for $SPM = \phi$ and thus isolating one of the six spreads embedded in $\mathcal{S}_4$, displays a binary quality: its vertices either belong to a single connected cluster or define isolated singletons.

Applying the Appendix A.1 definition for Ψ, the graph Laplacian for this example has characteristic equation

$$(\mathcal{S}_4)\ \ |\Psi - \kappa I| = \begin{vmatrix} \phi_1\phi_2 - \kappa & -\phi_1\phi_2 & 0 & 0 \\ -\phi_1\phi_2 & \phi_1\phi_2 - \kappa & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & -\kappa \end{vmatrix} = \kappa^3\left(\kappa - 2\phi_1\phi_2\right) = 0,$$

and its ranked spectrum is therefore $\Lambda_L(\kappa) = \{\kappa_1 = \kappa_2 = \kappa_3 = 0, \kappa_4 = 2\phi_1\phi_2 > 0\}$.

The following result (see Appendix A.5) generalizes this nexus between graph connectivity and the Laplacian spectrum. Because it explores the impact of measure degeneracy, no restrictions apart from nonnegativity and the adding-up constraint are imposed here on $SPM = \phi$.

Theorem 6 *Risk correction graph and Laplacian spectrum: Let $\mathcal{G}_\phi$ represent the risk correction graph with n vertices, corresponding to the n payoffs and to the n subjective probabilities ϕ_s associated with each state s, and with m edges, corresponding to the $m(n) \equiv \binom{n}{2} = n(n-1)/2$ spreads, each having the possibly vanishing probability-induced weight $\phi_i \phi_j \in [0,1], i \neq j$, defining atomistic risk for that spread. Then:*

(i) each vanishing probability induces a singleton component in respect of the associated vertex;

(ii) the non-vanishing probabilities induce a graph cluster gc with Laplacian $L_{gc}: (n - n_\phi) \times m_\phi$ where n_ϕ is the number of vanishing probabilities and $m_\phi = \binom{n - n_\phi}{2}$ is the number of weighted edges induced by the nonvanishing probabilities;

(iii) the total number of components in $\mathcal{G}_\phi$ is a linear function of the number of vanishing probabilities: $c = n_\phi + 1$;

(iv) the Laplacian for the graph cluster has rank $\rho[L_{gc}] = n - c$;

(v) each vertex in that cluster has degree $d(v) = n - c$;

(vi) the Laplacian L for $\mathcal{G}_\phi$ is a singular, symmetric, positive semidefinite operator; and

(vii) arranged in ascending order, the spectrum $\Lambda_L(\kappa)$ of the graph Laplacian L comprises real nonnegative eigenvalues, with its first c eigenvalues vanishing: $\kappa_1 = \ldots = \kappa_c = 0$.

As the example demonstrates, the edge weighting pattern imposed by the subjective measure gives the risk correction graph a binary quality under which each vertex either belongs to the single graph cluster or represents an isolated singleton. That binary separation and the properties of the graph cluster are linearly related to the number n_ϕ of vanishing probabilities. The number of sub-graphs increases one-for-one with n_ϕ as do the nullity of the Laplacian and the number of zero eigenvalues in its spectrum. Requiring $SPM = \phi$ to be strongly nondegenerate ($\phi > 0$) ensures that $\mathcal{G}_\phi$ is completely connected (comprises one cluster); that each vertex has degree $d(v) = n - 1$; and that the $n \times n$ graph Laplacian $\Psi = B^T \Pi B$ is a singular, positive semidefinite matrix of rank $n - 1$ having only one vanishing eigenvalue and $n - 1$ positive eigenvalues.

Distinctive spectral properties, unable to be addressed here (see Preston and Preston (2005b)), are associated with the risk-corrected and risk-adjusted rule sets. Directly or indirectly, both sets of results stem from the spectral properties of the graph Laplacian specified by Theorem 6. Our immediate interest refers to the property that as the subjective measure progressively degenerates, the number of sub-graphs $c = n_\phi + 1$ increases one-for-one with the number n_ϕ of vanishing probabilities and so therefore does the nullity c of the graph Laplacian. The dimensionality of the nullspace of the graph Laplacian is intimately related to the orientation of the risk correction partitioning of wealth space now to be defined.

6 Risk correction partitioning as analogue to spreads space

The isomorphism between SMVR/GLM as defined on spreads space and the rest of the risk-corrected and risk-adjusted rule set, defined only on wealth space, demands a conceptual counterpart to spreads space. That wealth space analogue for spreads space is the *risk correction partitioning*.

6.1 Partitionings of wealth space

The risk correction partitioning represents a *partitioning of wealth space*, which defines the *quotient space* (see Simmons (1963), pp 191-196) induced by some equivalence relationship. Three such equivalence relationships – assets having the same expected return under the subjective measure $SPM = \phi$, assets having the same price under the risk neutral measure $RNM = \pi$ and assets having the same risk correction under the risk pricing operator given by the measure differential $\phi - \pi$ – are indispensable to analysis of isomorphic risk-corrected and risk-adjusted rules.

Each partitioning has the same structure: the set of assets having zero value under a specific pricing operator constitutes the orthogonal complement to that operator (a linear subspace); and the sets of assets having the same positive value (in this context, pricing component) under the operator constitute hyperplanes parallel to that zero value subspace, which thereby partition wealth space in terms of the equivalence relationship implied by the pricing operator. We designate the partitioning of wealth space under the subjective measure as the *expected return partitioning*; under the risk neutral measure, as the *wealth shadow pricing partitioning*; and under the measure differential, as the *risk correction partitioning.*

6.2 Risk pricing operator

Comparison of the spreads mean-variance rule (12) and the compact graph Laplacian metamodel (20) demonstrates that multistate risk correction may be explained isomorphically in terms of risk correction functionals

$$rc = \Psi f^* \cdot x = \mu \cdot \Delta x \tag{23}$$

defined on wealth space and spreads space respectively. Applying (10) with Theorem 5, extend the risk correction isomorphisms defined on wealth space as

$$rc \equiv \lambda COV_\phi \left[x, W\right] = (\phi - \pi) \cdot x = \Psi f^* \cdot x = \Sigma f \cdot x, \tag{24}$$

by substituting the RNP rule (1) into the CAPM rule (4). Hence the implied risk pricing operator satisfies the isomorphic relationships:

$$\textit{Risk pricing operator: } \gamma \equiv \phi - \pi = \Psi f^* = \Sigma f \in \mathcal{W}^n \qquad (f^* \equiv \lambda W;\; f \equiv B f^*)\,. \tag{25}$$

Each of these three specifications plays a role in the sequel: one in this section, the other pair in the next section. Thus $\gamma = \Psi f^*$ represents the direct nexus from the graph Laplacian metamodel on spreads space to the risk correction partitioning as wealth-space analogue; $\gamma = \phi - \pi$ enables inferences to be made about the nexus between satisfaction of the non-arbitrage condition (equivalently, about existence of a risk neutral measure) and properties of the risk pricing operator; and $\gamma = \Sigma f$ defines the nexus between the risk pricing operator and the risk pricing invariants, the atomistic risks embedded in Σ and their shadow prices f, allowing inferences about shadow pricing bounds.

This risk pricing operator is now shown to *generate* the partitioning of wealth space into classes of equivalent risk assets. Whereas the atomistic risk corrector $\mu \equiv f\alpha \in \mathcal{D}^{m(n)}$ appearing in (23) expressly identifies the risk pricing invariants, its analogue, the risk pricing operator $\Psi f^* \in \mathcal{W}^n$, necessarily transforms and conceals that spreads-based information. As counterpart to the tripartite structure of the graph Laplacian, unravelling those transformations proves instructive for understanding spreads-based pricing more deeply.

6.3 Two sources of zero covariation

The risk pricing operator attributes a zero risk correction to two classes of assets – the first *structurally invariant* and the second *structurally dependent*. The safe asset cone SAC^n (7) defines the set of assets having zero covariation with the market portfolio whatever the choice of market portfolio, unit price of risk and subjective measure, so that

$$COV_\phi [S, W] = W^T \Psi S = \Delta W^T \Pi (\Delta S = 0) = 0 \qquad (S \in SAC^n).$$

Under the payoffs-based covariance specification, the zero sum operator Ψ annihilates all safe assets $S \in SAC^n$; under the spreads-based specification, $S \in SAC^n$ corresponds to the origin of spreads space and hence implies zero covariation. Thus absence of covariation between safe assets and the market portfolio is a structural invariant enforced for all (ϕ, W).

But for all $n > 2$ there is invariably also a structurally-dependent subset of risky assets exhibiting zero covariation with the market portfolio, distinguished in the following pair of definitions for specific ($\hat{\phi}$, $\hat{W}$) or universal choices (ϕ, W) of pricing benchmark:

$$IR^n \equiv \left\{ X \neq S \mid COV_{\widehat{\phi}} \left[X, \widehat{W} \right] = 0 \right\}; \qquad SAC^n \equiv \{ X \mid COV_\phi [X, W] = 0 \ \forall \ \phi, \ W \}.$$

Generated from spread interactions, such idiosyncratic risks are intrinsically multidimensional as the definition $rc = \sum_{k=1}^{m(n)} sc(k) = 0$ clarifies (Section 4.2): no such interactions are possible in the one-dimensional spreads space $\mathcal{D}^{m(n)=1}$ for $\mathcal{S}_2$.

The distinctive feature of idiosyncratic risks is that their existence, unlike safe assets, is structurally dependent on the direction or orientation of the risk pricing operator (now viewed as a geometric object). Risks that may be idiosyncratic under one direction set by the CAPM market portfolio and subjective measure lose that property under other directions. Better capturing that geometric sense of direction or orientation, idiosyncratic risks may be alternatively defined as:

$$(\mathcal{S}_n | n > 2) \quad \textit{Idiosyncratic risks:} \quad IR^n \equiv \{\gamma \cdot X = 0 \mid \ X \neq S\}.$$

With their risk not priced by the market, such assets are priced at their expected return – compare, for example, Cochrane (2002), pp. 17-18. By contrast, the complementary class of all other risky assets represents *systematic risks* that do attract a risk correction.

6.4 Equivalent risk assets

Systematic and *idiosyncratic risks* distinguish between risky assets that do, and do not, receive risk corrections in their pricing. *Equivalent risks* distinguish assets, whether risk-free or risky, that receive the same risk correction from those that do not. Thus idiosyncratic risks and risk-free assets jointly define the subspace of equivalent risk assets receiving no risk correction. Characterize that orthogonal complement to the risk pricing operator in terms of safe assets and idiosyncratic risks as follows (see Appendix A.6).

Theorem 7 *Risk correction partitioning: (i) Sources of zero covariation: the zero risk correction frontier in wealth space, $ZRCF^n$, is the $n-1$ dimensional subspace*

$$ZRCF^n \equiv IR^n \cup SAC^n \equiv \{X \mid \gamma \cdot X = 0\}$$

comprising the union of idiosyncratic risks IR^n and safe assets SAC^n. (ii) Structural dependence and invariance: idiosyncratic risks do not have invariant status under variation in the risk pricing operator or, equivalently, in the configuration of the market portfolio pricing benchmark (structural dependence), whereas safe assets will receive a zero risk correction under all such configurations (structural invariance). (iii) Equivalent risk assets: the assets $x, y \in RCF^n \equiv \{X \mid \gamma \cdot X = rc^*\}$ are equivalent risk assets with $y = x + S + IR$, so that any pair of equivalent risks differ at most by a position comprising a safe asset and an idiosyncratic risk, implying that $y - x \in ZRCF^n$. (iv) Idiosyncratic risks depend on atomistic risks: when all $m(n)$ atomistic risks are positive as a result of the strong condition $(SPM = \phi > 0)$ for nontrivial risk, the nullspace for the graph Laplacian,

$$(n_\phi = 0) \qquad N[\Psi] \equiv SAC^n \subset N\left[f^{*T}\Psi\right] \equiv ZRCF^n,$$

represents a one-dimensional subspace of the $n-1$ *dimensional zero risk correction frontier; but when all atomistic risks bar one vanish, as permitted under the weak condition* ($\phi_i \neq 1$)*, those nullspaces coincide, so that*

$$(n_\phi = n-2) \qquad SAC^n \subset N[\Psi] = N\left[f^{*T}\Psi\right] \equiv ZRCF^n.$$

A task for Section 7, result (iv) signals that much yet still remains to be extracted from the role of the risk correction partitioning, including the impact of $SPM = \phi$ on the characterization of idiosyncratic risks. But to apply the spectral theorem deriving from the risk correction graph (Section 5), when $n_\phi = n-2$ probabilities vanish, then the graph Laplacian Ψ has nullity $c = n_\phi + 1 = n-1$ to that $N[\Psi] = N\left[f^{*T}\Psi\right]$ has dimension $n-1$. Because the risk pricing operator always spans a one-dimensional subspace and always possesses an orthogonal complement of dimension $n-1$, at issue is its orientation. As n_ϕ increases progressively from zero (strong condition) to $n-2$ (extreme weak condition), that orientation changes: changing also is the delineation of which risks are idiosyncratic and which systematic.

7 Bounded shadow pricing of atomistic risks

Orthogonal to each of its equivalent risk hyperplanes, the risk pricing operator *generates* the risk correction partitioning induced on $\mathcal{W}^n$. The specific zero sum orientation of that generator is key to understanding analogues on wealth space of spreads-based pricing. We explore the properties of the risk pricing operator in four stages. The first stage defines a rebalancing operator embedded in the risk pricing operator that transforms shadow prices defined on spreads space into composite shadow prices defined on wealth space. The second stage examines relevant properties of those composite shadow prices. The third stage adduces the properties of the risk pricing operator as modified properties of the composite shadow prices. The fourth stage utilizes these properties to establish the bounded shadow pricing of atomistic risks.

7.1 Rebalancing operator

Spreads-based pricing underscores the role of zero sum operators. In fact, the risk pricing operator is zero sum (the graph Laplacian Ψ being zero sum, so necessarily is the linear combination Ψf^* – equivalently, the measure differential $\phi - \pi$ is zero sum). That property proves to have widespread ramifications. Consider from (25) $\Sigma f = \phi - \pi$. Its elements are a function of the vector shadow price of atomistic risk (which we seek to bound) and of the signed pairings of atomistic risks captured by the zero column sum operator Σ (whose transpose is defined by Theorem 2 (iv) as illustrated for $\mathcal{S}_4$ in Table 6 in Appendix A.3). Thus all column vectors in

$$(\mathcal{S}_n) \quad \Sigma_1 f(1) + \Sigma_2 f(2) + ... + \Sigma_m f(m) = \phi - \pi \tag{26}$$

are zero sum operators, as illustrated below for $\mathcal{S}_4$:

$$(\mathcal{S}_4) \begin{bmatrix} -\phi_1\phi_2 & -\phi_1\phi_3 & -\phi_1\phi_4 & 0 & 0 & 0 \\ \phi_1\phi_2 & 0 & 0 & -\phi_2\phi_3 & -\phi_2\phi_4 & 0 \\ 0 & \phi_1\phi_3 & 0 & \phi_2\phi_3 & 0 & -\phi_3\phi_4 \\ 0 & 0 & \phi_1\phi_4 & 0 & \phi_2\phi_4 & \phi_3\phi_4 \end{bmatrix} f = \begin{bmatrix} \phi_1 - \pi_1 \\ \phi_2 - \pi_2 \\ \phi_3 - \pi_3 \\ \phi_4 - \pi_4 \end{bmatrix}. \tag{27}$$

These zero sum operators distribute risk corrections consistently, or with zero sum, across each spread.

Now $\gamma = \Sigma f \in \mathcal{W}^n$ possesses generic structure. Factor Σf for $\mathcal{S}_4$ as $\Phi\Gamma f$ shown in Table 5. To evaluate the vector Γf generically or independently of a choice of n, add the nonzero elements of the corresponding *row* of Γ (described as the *rebalancing operator* in Definition 8) weighting each by the relevant shadow price f_{ji} determined by the *column* corresponding to each nonzero element (see first property now following).

Definition 8 *Generic structure of rebalancing operator* Γ*: For S_n, the factorization $\gamma = \Phi\Gamma f = \Sigma f$ necessarily implied by the definition for Σ (see Theorem 2 (iv), Section 2.3, and Table 6 in Appendix A.3) completely defines the $n \times m$ rebalancing operator Γ in terms of two properties: (i) the column order of Γ matches the spreads convention, as identified by the paired subscripts for the nonvanishing elements of each column; and (ii) in each column of Γ, the two probabilities for that $(j,i)'th$ spread appear with opposed sign and in reversed natural order – the $j'th$ probability appears in the $i'th$ row with negative sign and the $i'th$ probability appears in the $j'th$ row with positive sign.*

$$
(\mathcal{S}_4) \quad \Sigma f \equiv \begin{bmatrix} -\phi_1\phi_2 & -\phi_1\phi_3 & -\phi_1\phi_4 & 0 & 0 & 0 \\ \phi_1\phi_2 & 0 & 0 & -\phi_2\phi_3 & -\phi_2\phi_4 & 0 \\ 0 & \phi_1\phi_3 & 0 & \phi_2\phi_3 & 0 & -\phi_3\phi_4 \\ 0 & 0 & \phi_1\phi_4 & 0 & \phi_2\phi_4 & \phi_3\phi_4 \end{bmatrix} f
$$

$$
= \begin{bmatrix} \phi_1 & 0 & 0 & 0 \\ 0 & \phi_2 & 0 & 0 \\ 0 & 0 & \phi_3 & 0 \\ 0 & 0 & 0 & \phi_4 \end{bmatrix} \begin{bmatrix} -\phi_2 & -\phi_3 & -\phi_4 & 0 & 0 & 0 \\ \phi_1 & 0 & 0 & -\phi_3 & -\phi_4 & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & -\phi_4 \\ 0 & 0 & \phi_1 & 0 & \phi_2 & \phi_3 \end{bmatrix} f =
$$

$$
\equiv \Phi\Gamma f = \begin{bmatrix} \phi_1 & 0 & 0 & 0 \\ 0 & \phi_2 & 0 & 0 \\ 0 & 0 & \phi_3 & 0 \\ 0 & 0 & 0 & \phi_4 \end{bmatrix} \begin{bmatrix} -\phi_2 f_{21} - \phi_3 f_{31} - \phi_4 f_{41} \\ \phi_1 f_{21} - \phi_3 f_{32} - \phi_4 f_{42} \\ \phi_1 f_{31} + \phi_2 f_{32} - \phi_4 f_{43} \\ \phi_1 f_{41} + \phi_2 f_{42} + \phi_3 f_{43} \end{bmatrix} \equiv \Phi \left(\Gamma f \right).
$$

Table 5: *Factorization of risk pricing operator for* S_4

Under these generic properties for Γ, all probabilities appearing in its first row attract a negative sign; all probabilities appearing in its last row attract a positive sign; and the probabilities appearing in its intermediate rows attract mixed signs, but with the first always attracting a positive sign. Hence, as illustrated by Table 5, $\gamma = \Phi(\Gamma f)$ possesses recursive forward and backward structure. If $\phi_1 = 0$, its first element vanishes and, subject to that condition, all remaining probabilities in its second element attract negative signs; if $\phi_1 = \phi_2 = 0$, its first and second elements vanish and, subject to that condition, all remaining probabilities in its third element attract negative signs, and so on subject to some implicit stopping condition. If $\phi_n = 0$, its last element vanishes and, subject to that condition, all remaining probabilities in its second last element attract positive signs; and if $\phi_n = \phi_{n-1} = 0$, its last and second last elements vanish and, subject to that condition, all remaining probabilities in its third last element attract positive signs, and so on (again, subject to a stopping condition consistent with the forward recursion). Our immediate task is to capture the implications of that forward and backward recursion.

7.2 Composite shadow prices

In general, interaction of the rebalancing operator Γ with the shadow prices f produces sums of probability-weighted shadow prices $\Gamma f \in \mathcal{W}^n$ having distinctive sign patterns and related properties. The following theorem (see Appendix A.7) establishes that these *composite shadow prices* are bounded between their negative first element and positive last element; inherit the natural ordering of the market portfolio; and exhibit natural spreads representing the shadow prices for the natural spreads of the market portfolio.

Theorem 9 Properties of composite shadow prices $\Gamma f \in \mathcal{W}^n$: *(i)* Rebalancing of $(\Gamma f)_s$: *Let Γ be the $n \times m$ rebalancing operator with generic structure specified by Definition 8. Then the composite shadow prices $(\Gamma f)_s$, $s = 1, \ldots, n$, successively rebalance $n-1$ probability-weighted shadow prices of atomistic risk as the differential of backward-looking and forward-looking sums*

$$(\Gamma f)_s = \sum_{l=1}^{s-1} \phi_l f_{sl} - \sum_{l=s+1}^{n} \phi_l f_{ls}, \tag{28}$$

for the respective summations defined as zero when $s = 1$ and $s = n$. (ii) Bounded rebalancing: *All composite shadow prices satisfy the bounds*

$$-\left| (\Gamma f)_1 = \sum_{l=2}^{n} \phi_l f_{l1} \right| \leq (\Gamma f)_s \leq (\Gamma f)_n = \sum_{l=1}^{n-1} \phi_l f_{nl} \tag{29}$$

so that Γf invariably has its first element negative and its last element positive. (iii) Natural ordering of Γf: *Anchored between negative first and positive last elements, Γf inherits the natural ordering of the market portfolio. Its natural spreads, as implied by the definition (8), represent the shadow prices corresponding to the natural spreads for the market portfolio:*

$$(\Gamma f)_{s+1} - (\Gamma f)_s = f_{(s+1)s} > 0 \qquad (s = 1, \ldots, n-1)\,. \tag{30}$$

Hence, where a composite shadow price is negative or zero (positive), all preceding (succeeding) elements are negative (positive). (iv) Degrees of freedom: *Under rebalancing, at most one composite shadow price can vanish so that $(\Gamma f)_s = 0$ for at most one $s \in \{2, \ldots, n-1\}$.*

7.3 Risk pricing operator

In turn, because $\gamma = \Phi\Gamma f = \Sigma f$, the risk pricing operator inherits the structure and properties of the composite shadow prices – but modified by the specification of $SPM = \phi$

reflected in the subjective probability operator Φ.[1] The following result (see Appendix A.8) establishes properties of the risk pricing operator for the multistate model and is a crucial precursor to Theorem 14.

Theorem 10 Risk pricing operator $\gamma = \Phi\Gamma f = \Sigma f$: *(i)* Zero sum property of γ: *The risk pricing operator $\gamma = \Sigma f \in W^n$ is a zero sum operator. (ii)* Generic element γ_s: *Factored as $\gamma = \Phi\Gamma f$, the successive elements γ_s, $s = 1, \ldots, n$, of the risk pricing operator rebalance the differential between backward-looking and forward-looking sums of $n-1$ atomistic spread corrections, so that*

$$\gamma \equiv (\gamma_s) = \left(\sum_{l=1}^{s-1} \mu_{sl} - \sum_{l=s+1}^{n} \mu_{ls} \right) \equiv \left(\phi_s \sum_{l=1}^{s-1} \phi_l f_{sl} - \phi_s \sum_{l=s+1}^{n} \phi_l f_{ls} \right), \tag{31}$$

for the respective summations defined as zero when $s = 1$ and $s = n$. (iii) First and last nonvanishing elements: *$\gamma = \Phi\Gamma f$ exhibits first and last nonvanishing elements, $\gamma_F \neq 0$, $\gamma_L \neq 0$, that are respectively negative and positive:*

$$(\gamma_1 \leq 0; \gamma_1 = 0 \Rightarrow \gamma_2 \leq 0; \cdots; \gamma_F < 0); \ (\gamma_n \geq 0; \gamma_n = 0 \Rightarrow \gamma_{n-1} \geq 0; \cdots; \gamma_L > 0). \tag{32}$$

(iv) Natural ordering of γ: *The risk pricing operator possesses nonvanishing elements iff*

$$SPM = \phi > 0; \qquad \sum_{l=1}^{s-1} \phi_l f_{sl} \neq \sum_{l=s+1}^{n} \phi_l f_{ls} \quad (s = 2, \ldots, n-1), \tag{33}$$

in which case it inherits the natural ordering of the composite shadow prices if $\phi_{s+1} \leq \phi_s$ – or if $SPM = \phi$ is either inversely ordered (strong subjective pessimism) or uniformly distributed (subjective neutrality). (v) Vanishing expectation for composite shadow prices Γf: *Under $SPM = \phi$,*

$$\phi \cdot \Gamma f = \sum_{s=1}^{n} \phi_s (\Gamma f)_s = \sum_{s=1}^{n} \gamma_s = 0. \tag{34}$$

Explaining the nullspace composition result of Theorem 7 (iv), these results establish that the element γ_s vanishes if the relevant probability vanishes, $\phi_s = 0$, or if its backward and forward-looking terms balance (possible for one and only one internal element). The more of the former, the larger is $N[\Psi]$ relative to the $(n-1)$-dimensional subspace $N\left[f^{*T}\Psi\right]$, whose orientation but not dimensionality changes. Both features are idiosyncratic risk generators.

[1] Recalling Table 5 and the trivariate product $\Phi\Gamma f$, the element of ϕ belonging to any given row of Φ and those elements of ϕ belonging to the corresponding row of Γ partition $SPM = \phi$.

These collective properties have further implications pursued elsewhere. For immediate purposes, the rebalancing of backward-looking and forward-looking atomistic spread corrections reflected in (31) now permits the inference of bounded shadow pricing $\mathbf{0} < f \leq \mathbf{1}$ attaching to atomistic risks in $\mathcal{S}_n$.

7.4 Well-behaved pricing rules

Significant existence and uniqueness questions remain lurking in the base rule specification $p = MVR[\phi, \lambda, W;\ x, R]$. Resolving that indeterminacy will further restrict the positive shadow prices appearing as one factor of the risk pricing invariants $\mu \equiv f\alpha \in D^{m(n)}$. To specify the indeterminacy, frame the dependence reflected in the spreads mean-variance rule SMVR – namely, $Rp(x) = SMVR[\phi, f;\ x]$ – in the following manner. In the one-period context where discounting is (relatively) less interesting, suppress the discount rate by considering the forward rather than current price of an asset. Additionally, fixing asset payoffs legitimately focuses attention on the pricing of a specific asset. Allowing arbitrary specification of the nondegenerate subjective probability measure, as seems desirable given the absence of priors, then foists remaining indeterminacy onto the shadow price of atomistic risk.

These considerations motivate our formulation of the following existence question for asset pricing in $\mathcal{S}_n$.

Definition 11 Existence question: *Given the canonical asset x, do arbitrary nondegenerate choices of subjective measure $SPM = \phi$, under nontrivial risk $\{\lambda > 0,\ \Delta W > 0,\ \phi_i \neq 1\}$, imply admissible forward asset prices $Rp(x) = \pi \cdot x$ – corresponding to admissible risk neutral measures $RNM = \pi$ or equivalently to satisfaction of the non-arbitrage condition – for some choice of positive forward shadow prices $f \equiv \lambda \Delta W > 0$ applicable to atomistic risks $\alpha \in \mathcal{D}^{m(n)}$?*

Theorem 14 stipulates that global existence obtains for $\mathcal{S}_n$ if and only if the multistate shadow prices f are bounded on the positive unit interval. A further preliminary result will be needed to that end. Concentrate the probability mass of the subjective measure by turn on each spread. For indexing consistent with the spreads convention, define the related sets of *embedded spread measures*

$$\left\{\phi^{ji}\right\} \equiv \left(\phi_s \left|\phi_s = 0,\ s \neq \{i,\ j\}\,;\ \phi_i + \phi_j = 1,\ \phi_s \neq 1\right.\right). \tag{35}$$

Under nontriviality of risk (specifically, $\phi_s \neq 1$), these embedded sets of measures end-stop the progressive loss of connectivity in the risk correction graph under progressive degeneracy in the subjective measure. Figures 1 and 3 illustrate that loss of connectivity induced by $\phi \rightarrow \left\{\phi^{21}\right\}$.

The next result (see Appendix A.9) demonstrates that the *spread pricing operators* embedded in the risk pricing operator under the alternative sets of embedded spread measures (35) are naturally ordered irrespective of whether γ is ordered under less degenerate specifications for $SPM = \phi$ (per Theorem 10 (iv) for example).

Theorem 12 *Impact of measure degeneracy: (i) Nonvanishing equivalents: Under failure of the strong condition, let the sequenced probabilities $\phi(1), \ldots, \phi(\overline{n})$, $\overline{n} < n$, represent the $n - n_\phi$ nonvanishing probabilities of $SPM = \phi$. Then Theorem 10 applies in respect of appropriately reformulated expressions for nonvanishing elements. (ii) Embedded spread pricing operators: Hence the $m(n)$ sets of spread measures (35) imply the $m(n)$ embedded zero sum spread pricing operators*

$$\gamma_i = -\phi_i \phi_j f_{ji} \equiv -\mu_{ji}; \qquad \gamma_j = \phi_j \phi_i f_{ji} \equiv \mu_{ji}; \qquad \gamma_s = 0 \ \ (s \neq \{i, j\})\,; \qquad (j > i)\,. \tag{36}$$

(iii) Natural ordering of spread pricing operators: In contrast to Theorem 10 (iv), these embedded operators are invariably naturally ordered without regard to the embedded measures.

For the choices $\left\{\phi^{ji}\right\}$ of embedded measures, the $m(n)$ spread pricing operators (36) comprise zero sum operators with the *specific orientations* ($\phi_i - \pi_i < 0,\ \phi_j - \pi_j > 0$). These one-sided gradient restrictions $\nabla\phi > \nabla\pi$ on the embedded measures ensure that asset spreads positively correlated with the matching spread of the market portfolio attract a positive risk correction (see Appendix A.10 for the following result).

Theorem 13 *One-sided spreads-based gradient restrictions: Under nontriviality of risk, the asset risk correction conditioned on the embedded measures $\phi \rightarrow \left\{\phi^{ji}\right\}$ is positive iff the relevant spread correction is positive or iff*

$$rc = sc_{ji} = COV_{\phi \rightarrow \left\{\phi^{ji}\right\}}\left[x, W\right] > 0 \qquad \Leftrightarrow \qquad \nabla\phi^{ji} > \nabla\pi^{ji}.$$

Accordingly, these one-sided gradient restrictions must apply for each embedded spread pricing operator if $\gamma \equiv \phi - \pi$ is to qualify as a risk pricing operator. These various properties of the risk pricing operator enable the inference in Theorem 14 of bounded shadow pricing of atomistic risks $\mathbf{0} < f \leq \mathbf{1}$ for well-behaved pricing rules. For necessity of $f \leq \mathbf{1}$ (see Appendix A.11), choose $\phi \to \left\{\phi^{n1}\right\}$ as worst-case since the natural ordering on the market portfolio implies that $Max\ \{f\} = f_{n1}$. The necessity proof demonstrates that risk neutral measures $\{\pi^{n1}\}$ corresponding to the embedded spread measures $\phi \to \left\{\phi^{n1}\right\}$ exist only if $f_{n1} \leq 1$, whence $f \leq \mathbf{1}$ is necessary. The sufficiency proof starts with (31), establishing under any admissible choice of $SPM = \phi$ that $f \leq \mathbf{1}$ means that γ is oriented inside the unit ball so that $RNM = \pi$ always exists, given such choice of $SPM = \phi$, as $\pi = \phi - \gamma$ thereby satisfying Definition 11.

Theorem 14 Bounded shadow prices: *(i)* Unit bounds: *Definition 11 is satisfied if and only if the shadow prices for atomistic risks are bounded between zero and unity:*

$$(\mathcal{S}_n) \quad \mathbf{0} < f \equiv \lambda \Delta W \leq \mathbf{1}. \tag{37}$$

(ii) Characteristic scale of risk pricing invariants: *hence, for $SPM = \phi > 0$, atomistic spread corrections satisfy the implied bounds:*

$$(\mathcal{S}_n) \quad 0 < \mu(k) \equiv a(k) f(k) \leq 1/4, \quad k = 1, \ldots, m(n). \tag{38}$$

8 Commentary

Analyzing the pricing of risk on spreads space instead of conventionally on wealth space entails the sequencing of key steps: representation with the aid of zero sum operators of asset covariances as bilinear forms in asset spreads rather than asset payoffs; identification of the spreads mean-variance rule SMVR as a further risk-corrected rule isomorphic to CAPM as the base rule; recognition of the centrality of zero sum operators to the graph-theoretic specification of Strang's *Framework for Applied Mathematics*, representing a multidisciplinary and unifying modelling framework; our specialization of that framework as the graph Laplacian metamodel for risk correction; under nontriviality of risk, isolation of the atomistic *spread* corrections $\mu \equiv f\alpha \in \mathcal{D}^{m(n)}$ as risk pricing invariants demanding a comprehensive explanation; to that end, the matching of the intuitive geometry of the risk correction graph $\mathcal{G}_\phi$ to the algebraic properties of the zero sum graph Laplacian $\Psi = B^T \Pi B$, with degrading connectivity of that graph systematically reflecting the progressive degeneration of the subjective measure into the embedded spread measures $\phi \to \{\phi^{ji}\}$ in multiple ways over the paths of that graph; as wealth space analogue for spreads space, the complementary identification of the risk correction partitioning generated by the risk pricing operator $\gamma = \phi - \pi$ as analogue for the atomistic risk corrector $\mu \equiv f\alpha$ defined on spreads space; characterization of the risk pricing operator as a zero sum operator specifically oriented within the unit ball, having embedded spread pricing operators for each of the $m(n)$ spreads that are invariably naturally ordered; and inference from the successive rebalancing of atomistic spread corrections defining the risk pricing operator of unit bounds on the shadow prices of atomistic risk – all leading to provision of a five-factor spreads-based account of risk correction that identifies the respective roles of spread risks, their bounded shadow prices determined from the weighted market spreads, and spread scales, signs and interactions.

As a comparative statics exercise, select $SPM = \phi$ from the set of embedded spread measures $\phi \to \{\phi^{n1}\}$. This fixes the canonical asset's expected return $\phi \cdot x$ and also bounds its forward price under the non-arbitrage condition between the asset payoffs matching the dominant $(n, 1)'th$ spread of the market portfolio. For fixed expected return, variation in the admissible asset price will require, under a positive spread correlation with that market spread, inverse variation in the spread risk correction and hence in the dominant shadow price $f_{n1} \equiv \lambda \Delta W_{n1}$. The higher the admissible asset price in respect of that spread, the lower must be the shadow price of risk for that spread: however, a maximal asset price in respect of the dominant spread would require a vanishing shadow price, implying either a vanishing market spread or vanishing unit risk premium in contravention of the nontriviality of risk. Subject to their exclusion, fixing in this way an admissible asset price and embedded subjective measure fixes the bounded shadow price f_{n1} but, reflecting the parametric indeterminacy associated with CAPM, of itself determines neither of its factors λ and ΔW_{n1}. Hence smaller values for this dominant market spread, on that account driving the market portfolio (including, as necessary, other market spreads as well) towards the safe asset cone and reduced uncertainty in the pricing benchmark, will correspond to higher values of the unit risk premium applicable to all spreads – reflecting the higher degrees of risk aversion required for such choices of market portfolio to be optimal. Thus the spreads-based perspective on asset pricing leads to a distinctive alternative interpretation of the determination of the CAPM risk premium in terms of the risk pricing invariants $\mu \equiv f\alpha \in \mathcal{D}^{m(n)}$.

Priorities and space limitations mean that some issues desirably requiring further attention have not been addressed in this paper. Chief amongst these are isomorphisms amongst risk-corrected and risk-adjusted rules, the specification of and nexus between asset pricing metamodels, and potential queries about the definition of atomistic risk.

Nothwithstanding various isomorphisms utilized here, missing entirely from this paper are the risk-adjusted analogues for the multistate model that would anchor our multistate results to the prevailing asset pricing paradigm. In principle those analogues derive from the isomorphism $\pi = \phi - \Psi f^*$ between the risk neutral pricing rule (1) and the compact graph Laplacian metamodel (20) also assisted by the composite shadow prices, which have a specific role in the risk-adjusted rule set. Moreover, just as the graph Laplacian operator supports an asset pricing metamodel casting an umbrella naturally over risk-corrected pricing rules, on our interpretation Cochrane's *stochastic discount factor/generalized method of moments framework* similarly embraces risk-adjusted rules, with the two metamodels raising issues about their interaction. Preston and Preston (2005b) addresses both issues, applying spectral properties of the graph Laplacian to resolve risk-adjusted analogues for the shadow pricing of atomistic risks in the multistate model.

Under the graph Laplacian metamodel, atomistic risks receive a natural definition as the lowest level constituent of risk – namely, the absolute value of the unit state covariance between each pair of Arrow-Debreu assets. Collective expectations reflected in the subjective probability measure thus unavoidably intrude a subjective element into that definition of risk, a definition which applies equally to risk-adjusted rules. Preston and Preston (2005a) combines an alternative definition of unit risk purely in terms of the payoffs to the

Arrow-Debreu assets, together with geometric algebra as a mathematical language expressly designed to unify the modelling of the physical sciences, to derive a further risk-corrected rule. That *geometric pricing rule* mimics CAPM by pricing assets as a risk-free portfolio comprising an expected return bond offset by a risk correction bond liability.

A Appendix: Proofs

A.1 Covariance payoffs operator

$$\Psi = \Psi^T = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \phi_n \end{bmatrix} - \begin{bmatrix} \phi_1^2 & \phi_1\phi_2 & \cdots & \phi_1\phi_n \\ \phi_1\phi_2 & \phi_2^2 & \cdots & \phi_2\phi_n \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1\phi_n & \phi_2\phi_n & \cdots & \phi_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1(1-\phi_1) & -\phi_1\phi_2 & \cdots & -\phi_1\phi_n \\ -\phi_1\phi_2 & \phi_2(1-\phi_2) & \cdots & -\phi_2\phi_n \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_1\phi_n & -\phi_2\phi_n & \cdots & \phi_n(1-\phi_n) \end{bmatrix} \equiv \Phi - \phi\phi^T$$

A.2 Definition 1: Spreads convention

The definition for $\mathcal{S}_4$ illustrates the spreads convention:

$$(\mathcal{S}_4) \begin{bmatrix} \Delta x_{21} \\ \Delta x_{31} \\ \Delta x_{41} \\ \Delta x_{32} \\ \Delta x_{42} \\ \Delta x_{43} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{39}$$

Let $\Delta x_{ji} \equiv B(k)\,x$ for $j = 2, \ldots, n$; $i = 1, \ldots, n-1$; $j > i$; and for $B(k)$ the $k'th$ row of B, for $k = 1, \ldots, m$. The spreads convention $\Delta x_{ji} \equiv x_j - x_i$ $(j > i;\ j$ *iterated first*$)$ implies that the rows of B,

$$B(k) = \left[\begin{array}{ccccc} \mathbf{0} & -1 & \mathbf{0} & 1 & \mathbf{0} \end{array}\right], \quad (k = 1, \ldots, m),$$

have at least one and at most three nonempty blocks of zeroes, and possess only an ordered pair of nonvanishing entries of $\mp 1$for their $i'th$ and $j'th$ columns respectively. Since each row $B(k)$ is zero sum, the differencing operator is a zero row sum operator. In general, B has rank $n-1$ reflecting the available degrees of freedom in specifying spreads, with its first $n-1$ columns summing to the negative of the $n'th$ column.

A.3 Theorem 2: Risk correction isomorphisms

First identity: $COV_\phi[x, W] = \left(W - \overline{W}\right)^T \Phi x$

$$COV_\phi[x, W] \ = \ E_\phi[xW] - E_\phi[x]\,E_\phi[W] = \sum \phi_s W_s x_s - \overline{W} \sum \phi_s x_s = \sum \phi_s x_s \left(W_s - \overline{W}\right),$$

for subjective probability operator $\Phi \equiv diag[\phi]$ and expected wealth $\overline{W} \equiv E_\phi[W]$.

Second identity: $\left(W - \overline{W}\right)^T \Phi = W^T \Psi$

$$W - \overline{W} = \begin{bmatrix} \overline{\phi}_1 & -\phi_2 & \cdots & -\phi_n \\ -\phi_1 & \overline{\phi}_2 & \cdots & -\phi_n \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_1 & -\phi_2 & \cdots & \overline{\phi}_n \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} \equiv PW,$$

so that $\left(W - \overline{W}\right)^T \Phi = W^T P^T \Phi$ and

$$P^T \Phi = \begin{bmatrix} \overline{\phi}_1 & -\phi_1 & \cdots & -\phi_1 \\ -\phi_2 & \overline{\phi}_2 & \cdots & -\phi_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_n & -\phi_n & \cdots & \overline{\phi}_n \end{bmatrix} \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \phi_n \end{bmatrix} = \Psi.$$

$$(\mathcal{S}_4)\ \begin{bmatrix} W_1 & W_2 & W_3 & W_4 \end{bmatrix} \begin{bmatrix} \phi_1\overline{\phi_1} & -\phi_2\phi_1 & -\phi_3\phi_1 & -\phi_4\phi_1 \\ -\phi_1\phi_2 & \phi_2\overline{\phi_2} & -\phi_3\phi_2 & -\phi_4\phi_2 \\ -\phi_1\phi_3 & -\phi_2\phi_3 & \phi_3\overline{\phi_3} & -\phi_4\phi_3 \\ -\phi_1\phi_4 & -\phi_2\phi_4 & -\phi_3\phi_4 & \phi_4\overline{\phi_4} \end{bmatrix} = W^T\Psi$$

$$= \begin{bmatrix} -\phi_1\phi_2 \triangle W_{21} - \phi_1\phi_3 \triangle W_{31} - \phi_1\phi_4 \triangle W_{41} \\ \phi_1\phi_2 \triangle W_{21} - \phi_2\phi_3 \triangle W_{32} - \phi_2\phi_4 \triangle W_{42} \\ \phi_1\phi_3 \triangle W_{31} + \phi_2\phi_3 \triangle W_{32} - \phi_3\phi_4 \triangle W_{43} \\ \phi_1\phi_4 \triangle W_{41} + \phi_2\phi_4 \triangle W_{42} + \phi_3\phi_4 \triangle W_{43} \end{bmatrix}^T$$

$$= \begin{bmatrix} \triangle W_{21} \\ \triangle W_{31} \\ \triangle W_{41} \\ \triangle W_{32} \\ \triangle W_{42} \\ \triangle W_{43} \end{bmatrix}^T \begin{bmatrix} -\phi_1\phi_2 & \phi_1\phi_2 & 0 & 0 \\ -\phi_1\phi_3 & 0 & \phi_1\phi_3 & 0 \\ -\phi_1\phi_4 & 0 & 0 & \phi_1\phi_4 \\ 0 & -\phi_2\phi_3 & \phi_2\phi_3 & 0 \\ 0 & -\phi_2\phi_4 & 0 & \phi_2\phi_4 \\ 0 & 0 & -\phi_3\phi_4 & \phi_3\phi_4 \end{bmatrix} = \triangle W^T\Sigma^T.$$

Table 6: *Mixed covariance operator for n=4 and m=6*

Third identity: $W^T\Psi = \triangle W^T\Sigma^T$ Table 6 writes the covariance payoffs operator Ψ, as defined in Appendix A.1, for $n = 4$ and with probability complements denoted by bars. It evaluates and then re-factors that product. Observe the interchangeability of the probability-weighted market payoffs $W^T\Psi$ and probability-weighted market spreads $\triangle W^T\Sigma^T$. For any choice of n apply the general definitions for Ψ, Σ^T and B. Hence $\triangle W = BW$ implies $W^T\Psi = \triangle W^T\Sigma^T = W^TB^T\Sigma^T$ so that $\Psi = B^T\Sigma^T = \Sigma B$.

Fourth identity: $\Sigma^T x = \Pi \triangle x$ The *mixed covariance operator* Σ^T illustrated by Table 6 inherits the zero row sum property of the covariance payoffs operator Ψ, thus operating on asset payoffs x to generate asset spreads weighted by the relevant probability products. Table 7 illustrates the final equality of (10). For any choice of n, apply the general definitions for Ψ, Σ and B again to supply $\triangle W^T\Sigma^T x = \triangle W^T\Pi\Delta x = \triangle W^T\Pi B x$ so that $\Sigma^T = \Pi B$. Hence, using the proof of the third identity, $\Psi = B^T\Sigma^T = B^T\Pi B$. Further, $\lambda\Pi\Delta W =$

$$(\mathcal{S}_4) \quad \begin{bmatrix} \triangle W_{21} \\ \triangle W_{31} \\ \triangle W_{41} \\ \triangle W_{32} \\ \triangle W_{42} \\ \triangle W_{43} \end{bmatrix}^T \begin{bmatrix} \phi_1\phi_2 \triangle x_{21} \\ \phi_1\phi_3 \triangle x_{31} \\ \phi_1\phi_4 \triangle x_{41} \\ \phi_2\phi_3 \triangle x_{32} \\ \phi_2\phi_4 \triangle x_{42} \\ \phi_3\phi_4 \triangle x_{43} \end{bmatrix} = \triangle W^T \Sigma^T x$$

$$= \begin{bmatrix} \triangle W_{21} \\ \triangle W_{31} \\ \triangle W_{41} \\ \triangle W_{32} \\ \triangle W_{42} \\ \triangle W_{43} \end{bmatrix}^T \begin{bmatrix} \phi_1\phi_2 & 0 & & \cdots & & 0 \\ 0 & \phi_1\phi_3 & & & & \\ & & \phi_1\phi_4 & \ddots & & \vdots \\ \vdots & & \ddots & \phi_2\phi_3 & & \\ & & & & \phi_2\phi_4 & 0 \\ 0 & & \cdots & & 0 & \phi_3\phi_4 \end{bmatrix} \begin{bmatrix} \triangle x_{21} \\ \triangle x_{31} \\ \triangle x_{41} \\ \triangle x_{32} \\ \triangle x_{42} \\ \triangle x_{43} \end{bmatrix}$$

$$= \triangle W^T \Pi \triangle x.$$

Table 7: *State covariance operator for n=4 and m=6*

$\lambda \Pi B W = \lambda \Sigma^T W$ (as used in (13)).

A.4 Equation (14): Unit state covariances

$$COV_\phi [\delta_i, \delta_j] = \delta_j^T \Psi \delta_i \qquad (j \neq i)\,.$$

Hence $\Psi\delta_i$ selects the i'th column of Ψ, and $\delta_j^T \Psi \delta_i$ accordingly selects the (j,i)'th element of Ψ. From Appendix A.1, $\delta_j^T \Psi \delta_i = -\phi_i\phi_j$, $j \neq i$. Thus, also using (11), (14) follows from:

$$COV_\phi [\delta_i, \delta_j] = \delta_j^T \Psi \delta_i = \delta_j^T B^T \Pi B \delta_i = \Delta\delta_j^T \Pi \Delta\delta_i = -\phi_i\phi_j, \qquad j \neq i.$$

A.5 Theorem 6: Risk correction graph and Laplacian spectrum

(i) $\phi_i = 0$ isolates vertex v_i from all other vertices, since all its edge weights vanish: $\phi_i\phi_j = 0$ $\forall\ j \neq i$;

(ii) because of the 'chaining' property of the edge weights $\phi_i\phi_j$, the $n - n_\phi$ remaining vertices necessarily form a completely connected sub-graph in $\mathcal{G}_\phi$ with $m_\phi = \binom{n - n_\phi}{2}$ edges;

(iii) that cluster plus the singleton components imply that the number of sub-graphs is $c = n_\phi + 1$;

(iv) by the zero row sum property of L_{gc}, $L_{gc}v = 0$ iff $v = c = Sp[\mathbf{1}]$ for $\mathbf{1}$ the vector of units. Hence the nullity of L_{gc} is $\eta[L_{gc}] = 1$, so that its rank is $\rho[L_{gc}] = n - n_\phi - 1 = n - c$;

(v) since there are $n - n_\phi$ vertices in that completely connected cluster, each vertex v is connected to $n - n_\phi - 1$ vertices and so has degree $d(v) = n - c$;

(vi) by the preceding result, the $n \times n$ Laplacian for $\mathcal{G}_\phi$ has rank $\rho[L] = \rho[L_{gc}] = n - c$ and nullity $\eta[L] = c$. Applying Gerschgorin's 'circle theorem' derived from the row components of $Lx - \kappa x = 0$ – see Strang (1976), p289, or Cremean and Murray (2003) – every eigenvalue of a matrix L lies in at least one of the circles C_i with center at the diagonal entry L_{ii} and its radius $r_i = \sum_{j \neq i} |L_{ij}|$ equal to the absolute sum along the rest of the row. But by the zero row sum property of the Laplacian, its Gerschgorin circles must have centres $\phi_i \overline{\phi}_i > 0$ and radii $\phi_i \overline{\phi}_i$, $i = 1, ..., n$. Hence the Laplacian eigenvalues are real (since $L = L^T$) and nonnegative (given these radii), with at least one zero by (vi) since $c \geq 1$ by (iii). Thus the Laplacian is a singular, symmetric positive semidefinite matrix since $x^T L x \geq 0$ with equality applying for the eigenvector $x = \mathbf{1}$ corresponding to the zero eigenvalue(s);

(vii) ranking the Laplacian spectrum in ascending order, its first c eigenvalues — corresponding to its nullity $\eta[L] = c$ — are therefore zero.

A.6 Theorem 7: Risk correction partitioning

(i) Sources of zero covariation: $\Psi f^* \cdot X = 0 \Leftrightarrow f^{*T} \Psi X = 0 \Leftrightarrow X \in N\left[f^{*T}\Psi\right] \equiv ZRCF^n$. But $\Psi S = 0$, $S \in SAC^n$, implies $SAC^n = N[\Psi] \subset \left(N\left[f^{*T}\Psi\right] \equiv ZRCF^n\right)$. SAC^n is one-dimensional; equivalently, since the graph Laplacian Ψ has rank $n-1$ under strongly nondegenerate $SPM = \phi > 0$ (see Theorem 6), its nullspace is also one-dimensional and thus $SAC^n = N[\Psi]$. But $N\left[f^{*T}\Psi\right] \equiv ZRCF^n$ is necessarily of dimension $n-1$ since it defines the hyperplane $\Psi f^* \cdot X = 0$. Hence define $IR^n \equiv \left\{N\left[f^{*T}\Psi\right] \mid X \neq S\right\}$ so that $ZRCF^n \equiv \{IR^n \cup SAC^n\} = N\left[f^{*T}\Psi\right]$.

(ii) Structural dependence and invariance: As the risk pricing operator Ψf^* varies with the CAPM configuration (W, ϕ), its direction necessarily changes and hence so does the orientation of its orthogonal complement $N\left[f^{*T}\Psi\right] \equiv ZRCF^n$. Those changing orthogonal complements can intersect only in subspaces of dimension $n-2$, a proper subspace of the $n-1$ dimensional $ZRCF^n$. Specifically, applying Kaplan and Lewis (1971), Theorem 14, p 675, if A and B are distinct subspaces $A \neq B$ of dimension $n-1$ of an n dimensional vector space, then

$$\dim(A \cap B) = \dim A + \dim B - \dim\{A + B\} = 2n - 2 - n = n - 2. \tag{40}$$

Allowing for the embedded one-dimensional safe asset cone SAC^n, idiosyncratic risks surviving respecification of the risk pricing operator must constitute a proper subset of that intersection subspace and one that varies under repeated respecifications. Hence idiosyncratic risks do not have invariant status, in general varying with (W, ϕ).

(iii) Equivalent risk assets: For risk correction index rc^*, let $x,\ y \in RCF^n \equiv \{X \mid \gamma \cdot X = rc^*\}$. Then their difference must belong to the parallel subspace: $y - x \in ZRCF^n \equiv \{IR^n \cup SAC^n\} \equiv N\left[f^{*T}\Psi\right]$. Hence, for some $S \in SAC^n$ and $IR \in \left\{N\left[f^{*T}\Psi\right] \mid X \neq S\right\}$,

$$f^{*T}\Psi[y - x] = f^{*T}\Psi[S + IR] = 0.$$

(iv) Impact of atomistic risks on idiosyncratic risks: (i) establishes $SAC^n = N[\Psi] \subset N\left[f^{*T}\Psi\right]$ for $\phi > 0$. From Appendix A.1, when only one atomistic risk survives, then Ψ has rank one, and $N[\Psi]$ has dimension $n-1$. The dimension of the zero risk correction frontier is unchanged, SAC^n remains structurally embedded in $ZRCF^n$ but the direction of the risk pricing operator Ψf^* has changed to provide a different set of idiosyncratic risks in response to the degeneration in $SPM = \phi$.

$$(\mathcal{S}_4)\quad \begin{bmatrix} 0 \\ \phi_1 f_{\mathbf{2}1} \\ \phi_1 f_{\mathbf{3}1} + \phi_2 f_{\mathbf{3}2} \\ \phi_1 f_{\mathbf{4}1} + \phi_2 f_{\mathbf{4}2} + \phi_3 f_{\mathbf{4}3} \end{bmatrix} - \begin{bmatrix} \phi_2 f_{2\mathbf{1}} + \phi_3 f_{3\mathbf{1}} + \phi_4 f_{4\mathbf{1}} \\ \phi_3 f_{3\mathbf{2}} + \phi_4 f_{4\mathbf{2}} \\ \phi_4 f_{4\mathbf{3}} \\ 0 \end{bmatrix} \equiv g - h$$

Table 8: *Vector difference illustrated for S_4*

A.7 Theorem 9: Properties of composite shadow prices

(i) Rebalancing of $(\Gamma f)_s$: In view of the mixed-sign terms in the elements of Γ, and thus of Γf, write the latter as the difference of two positive terms so that, illustrated by $\mathcal{S}_4$, the vector Γf becomes the vector difference shown in Table 8. The vectors g and h have the following properties. Firstly, for $s = 1, \ldots, n$, vector g possesses elements g_s that have $s-1$ nonvanishing terms; h possesses elements h_s that have $n-s$ nonvanishing terms. Second, in terms of f_{ji}, the rows of g are indexed over $j = 1, \ldots, n$ (bolded for identification in the $\mathcal{S}_4$ example), with vanishing element corresponding to $j = 1$ (consistent with the spreads convention, which indexes j over $2, \ldots, n$). Third, in terms of f_{ji}, the rows of h are indexed over $i = 1, \ldots, n$ (bolded for identification), with vanishing element corresponding to $i = n$ (consistent with the spreads convention, which indexes i over $1, \ldots, n-1$). Finally, noting that the order of $\{f_{ji}\}$ over h corresponds to the order of the spreads imposed by that convention, the pyramidal/inverted-pyramidal structure of g and h generalizes to $\mathcal{S}_n$ in straightforward fashion.

So generalized, express the $s'th$ element of $\Phi(\Gamma f)$ generically as

$$(\mathcal{S}_n)\quad \begin{cases} (\Gamma f)_s = g_s - h_s; \qquad (s = 1, \ldots, n) \\ \\ g \equiv (g_s) = \begin{bmatrix} \vdots \\ \sum_{i=1}^{s-1} \phi_i f_{si} \\ \vdots \end{bmatrix}; \qquad h \equiv (h_s) = \begin{bmatrix} \vdots \\ \sum_{j=s+1}^{n} \phi_j f_{js} \\ \vdots \end{bmatrix}. \end{cases} \tag{41}$$

Both summations (individual elements or composite shadow prices) have a maximum of $n-1$ terms (the first when $s = n$, the second when $s = 1$), consistent with the pyramidal

$$g = \begin{bmatrix} 0 \\ \sum_{i=1}^{\mathbf{2}-1} \phi_i f_{\mathbf{2}i} = \phi_1 f_{\mathbf{2}1} \\ \sum_{i=1}^{\mathbf{3}-1} \phi_i f_{\mathbf{3}i} = \phi_1 f_{\mathbf{3}1} + \phi_2 f_{\mathbf{3}2} \\ \sum_{i=1}^{\mathbf{4}-1} \phi_i f_{\mathbf{4}i} = \phi_1 f_{\mathbf{4}1} + \phi_2 f_{\mathbf{4}2} + \phi_3 f_{\mathbf{4}3} \end{bmatrix}; \quad h = \begin{bmatrix} \sum_{j=2}^{4} \phi_j f_{j\mathbf{1}} = \phi_2 f_{2\mathbf{1}} + \phi_3 f_{3\mathbf{1}} + \phi_4 f_{4\mathbf{1}} \\ \sum_{j=3}^{4} \phi_j f_{j\mathbf{2}} = \phi_3 f_{3\mathbf{2}} + \phi_4 f_{4\mathbf{2}} \\ \sum_{j=4}^{4} \phi_j f_{j\mathbf{3}} = \phi_4 f_{4\mathbf{3}} \\ 0 \end{bmatrix}$$

Table 9: *Generic elements illustrated for S_4*

and inverted-pyramidal structure illustrated by $\mathcal{S}_4$. The index $s = 1, \ldots, n$ is external to both summands and selects the same row elements of g and h. Both summations require the convention that when the upper index is less than the lower index, that element is defined as zero (hence $g_1 = h_n = 0$). Table 9 illustrates these expressions for $n = 4$. In particular, each choice of $s = 1, \ldots, 4$ (bolded for ease of identification) selects the same row element of g and h. The expression (28) follows by replacing the purely internal indices i and j in (41) with l.

(ii), (iii) Bounded rebalancing and natural ordering: Consider $s = 1$ in (28). Then $(\Gamma f)_1 = -\sum_{l=2}^{n} \phi_l f_{l1} < 0$ since the weak condition requires at least two nonvanishing probabilities. Similarly, for $s = n$, $(\Gamma f)_n = \sum_{l=1}^{n-1} \phi_l f_{nl} > 0$. Hence Γf possesses elements anchored beween a negative first element and a positive last element.

To establish natural ordering, consider the expressions for the $s'th$ and $(s+1)'th$ elements:

$$(\Gamma f)_s = \sum_{l=1}^{s-1} \phi_l f_{sl} - \sum_{l=s+1}^{n} \phi_l f_{ls}; \quad (\Gamma f)_{s+1} = \sum_{l=1}^{s} \phi_l f_{(s+1)l} - \sum_{l=s+2}^{n} \phi_l f_{l(s+1)}.$$

Then Table 10 establishes that all successive differences are positive under the natural spreads defining the natural ordering of the market portfolio. Table 11 illustrates for S_3. This ordering holds whether or not a particular element of Γf vanishes. If $h_s - g_s = 0$, then $h_{s-1} < h_s$ and $g_{s-1} > g_s$ imply that $h_{s-1} - g_{s-1} < 0$. Similarly, $h_{s+1} - g_{s+1} > 0$ if

$$
\begin{aligned}
&(\Gamma f)_{s+1} - (\Gamma f)_s = \\
&\sum_{l=1}^{s-1} \phi_l f_{(s+1)l} + \phi_s f_{(s+1)s} - \sum_{l=1}^{s-1} \phi_l f_{sl} - \sum_{l=s+2}^{n} \phi_l f_{l(s+1)} + \phi_{s+1} f_{(s+1)s} + \sum_{l=s+2}^{n} \phi_l f_{ls} \\
&= \sum_{l=1}^{s-1} \phi_l \left(f_{(s+1)l} - f_{sl}\right) + \sum_{l=s+2}^{n} \phi_l \left(f_{ls} - f_{l(s+1)}\right) + \left(\phi_s + \phi_{s+1}\right) f_{(s+1)s} \\
&= \sum_{l=1}^{s-1} \phi_l f_{(s+1)s} + \sum_{l=s+2}^{n} \phi_l f_{(s+1)s} + \left(\phi_s + \phi_{s+1}\right) f_{(s+1)s} \\
&= f_{(s+1)s} \sum_{l=1}^{n} \phi_l = f_{(s+1)s} > 0 \qquad (s = 1, \ldots, n-1)
\end{aligned}
$$

Table 10: *Natural ordering of* Γf

$$
(\mathcal{S}_3) \quad
\begin{aligned}
&g(3) - h(3) = \begin{bmatrix} 0 \\ \phi_1 f_{21} \\ \phi_1 f_{31} + \phi_2 f_{32} \end{bmatrix} - \begin{bmatrix} \phi_2 f_{21} + \phi_3 f_{31} \\ \phi_3 f_{32} \\ 0 \end{bmatrix} \\
&g_2 - h_2 + h_1 = \phi_1 f_{21} - \phi_3 f_{32} + \phi_2 f_{21} + \phi_3 f_{31} \\
&\quad = \phi_1 f_{21} + \phi_2 f_{21} + \phi_3 f_{21} = f_{21} > 0 \\
&g_3 - g_2 + h_2 = \phi_1 f_{31} + \phi_2 f_{32} - \phi_1 f_{21} + \phi_3 f_{32} \\
&\quad = \phi_1 f_{32} + \phi_2 f_{32} + \phi_3 f_{32} = f_{32} > 0
\end{aligned}
$$

Table 11: *Natural ordering of* Γf *for* S_3

$h_s - g_s = 0$. For example, let the middle element of Table 11 vanish (see (iv) for significance). The shadow prices $f_{(s+1)s} \equiv \lambda \Delta W_{(s+1)s} > 0$ are those associated with the natural spreads (8) of the market portfolio.

(iv) Degrees of freedom: To interpret $(\Gamma f)_s = 0$, suppose the strong condition $SPM = \phi > 0$ is satisfied under a uniform distribution. Hence, applying (28), simplify the balancing constraint as

$$
(\Gamma f)_s = \sum_{l=1}^{s-1} \phi_l f_{sl} - \sum_{l=s+1}^{n} \phi_l f_{ls} = \sum_{l=1}^{s-1} \Delta W_{sl} - \sum_{l=s+1}^{n} \Delta W_{ls} = 0 \quad (s \in \{2, \ldots, n-1\}),
$$

where $s = \{1, n\}$ are excluded by the conditions $h_1 - g_1 = h_1 < 0$ and $g_n - h_n = g_n > 0$. Under uniform weighting, this balancing constraint utilizes all $n-1$ degrees of freedom as previously specified in terms of (6): $s-1$ in respect of ΔW_{sl} and $n-s$ in respect of ΔW_{ls}. So only one of the internal elements of Γf can vanish in this way. If $(\Gamma f)_s = g_s - h_s = 0$, then dominance arguments based on comparisons of g_s with $g_{s\pm1}$ and of h_s with $h_{s\pm1}$ establish that neither $g_{s\pm1} - h_{s\pm1} = 0$. More than one such vanishing element would of course contradict the natural ordering established for Γf. Under this idiosyncratic risk generator $\gamma_s = 0$, payoffs x_s attract no risk correction because of the happenstance configuration of (equally weighted) market portfolio spreads.

A.8 Theorem 10: Risk pricing operator $\gamma = \Phi\Gamma f = \Sigma f$

(i) Zero sum property of γ: Σ is zero column sum. Hence the linear combination Σf is also zero sum.

(ii) Generic element γ_s: Applying (41) with $\gamma = \Phi\Gamma f$,

$$\gamma_s = \phi_s g_s - \phi_s h_s = \phi_s \sum_{l=1}^{s-1} \phi_l f_{sl} - \phi_s \sum_{l=s+1}^{n} \phi_l f_{ls} \equiv \sum_{l=1}^{s-1} \mu_{sl} - \sum_{l=s+1}^{n} \mu_{ls}$$

for atomistic spread corrections defined by $\mu_{ji} \equiv \phi_j \phi_i f_{ji}$.

(iii) First and last nonvanishing elements: Conditions (32) are established as follows. Applying (31),

$$\gamma_1 = -\phi_1 \sum_{l=2}^{n} \phi_l f_{l1} \Rightarrow (\gamma_1 \leq 0 \Leftrightarrow \phi_1 \geq 0)\,.$$

If $\phi_1 = 0$, then

$$\gamma_2 = -\phi_2 \sum_{l=3}^{n} \phi_l f_{l2} \Rightarrow (\gamma_2 \leq 0 \Leftrightarrow \phi_2 \geq 0)\,.$$

Nondegeneracy ($\phi_s \neq 1$) guarantees a choice of $\phi_{s'} > 0$ and at least one further $s'' > s'$ such that the first nonvanishing element (FNVE) satisfies

$$\textit{FNVE:}\ (\phi_{s'},\ \phi_{s''}\ > 0;\ s'' > s') \Rightarrow \gamma_{s'} = -\phi_{s'} \sum_{l=s'+1}^{n} \phi_l f_{ls'} < 0.$$

Similarly:

$$\gamma_n = \phi_n \sum_{l=1}^{n-1} \phi_l f_{nl}\ \Rightarrow (\gamma_n \geq 0 \Leftrightarrow \phi_n \geq 0)\,.$$

If $\phi_n = 0$, then

$$\gamma_{n-1} = \phi_{n-1} \sum_{l=1}^{n-2} \phi_l f_{(n-1)l} \ \Rightarrow \left(\gamma_{n-1} \geq 0 \Leftrightarrow \phi_{n-1} \geq 0\right).$$

Nondegeneracy guarantees a choice $s'' > s'$ such that the last nonvanishing element (LNVE) satisfies

$$\textit{LNVE:}\ \left(\phi_{s'},\ \phi_{s''}\ > 0;\ s'' > s'\right) \Rightarrow \gamma_{s''} = \phi_{s''} \sum_{l=1}^{s''-1} \phi_l f_{s''l}\ > 0.$$

(iv) Natural ordering of γ: The two conditions of (33) are obviously necessary and sufficient for (31) to have nonvanishing elements for $s = 1, \ldots, n$. (Because $g_1 = h_n = 0$, $SPM = \phi > 0$ implies that $g_s - h_s$ can vanish only for indices $s = 2, \ldots, n-1$.) From (31), the naturally ordered elements of (28) are successively weighted by ϕ_s to form γ. It is sufficient for that natural ordering to be preserved if $\phi_{s+1} \leq \phi_s$ or if $SPM = \phi$ is either inversely ordered or uniformly distributed.

(v) Vanishing expectation for composite shadow prices Γf: Since

$$\phi \cdot \Gamma f = \sum_{s=1}^{n} \phi_s \left(\Gamma f\right)_s = \sum_{s=1}^{n} \gamma_s = 0 \tag{42}$$

by the zero sum property of γ in conjunction with (31), Γf has a vanishing expectation under $SPM = \phi$. This property has significance for risk-adjusted rules – see Preston and Preston (2005b).

A.9 Theorem 12: Impact of measure degeneracy

(i) Nonvanishing equivalents: Let $\phi_{(1)}$, $\phi_{(2)}$, ..., $\phi_{(\overline{n})}$ denote the $\overline{n} = n - n_\phi < n$ nonvanishing probabilities taken in their embedded order in $SPM = \phi$. Let $\gamma_{(s)}$ be the element of γ corresponding to the nonvanishing probability $\phi_{(s)}$. Reformulate (31) as:

$$\gamma_{(s)} = \phi_{(s)} \left(\sum_{(l)=(1)}^{(s)-(1)} \phi_{(l)} f_{(s)(l)} - \sum_{(l)=(s)+(1)}^{(\overline{n})} \phi_{(l)} f_{(l)(s)} \right).$$

Table 12 evaluates individual elements.

$$
\begin{aligned}
&\gamma_{(1)} = -\phi_{(1)} \sum_{(l)=(1)+(1)}^{\overline{(n)}} \phi_{(l)} f_{(l)(1)} < 0 \\
&\gamma_{(2)} = \phi_{(2)} \left(\phi_{(1)} f_{(2)(1)} - \sum_{(l)=(2)+1}^{(\overline{n})} \phi_{(l)} f_{(l)(2)} \right) \\
&\gamma_{(3)} = \phi_{(3)} \left(\sum_{(l)=(1)}^{(3)-(1)} \phi_{(l)} f_{(3)(l)} - \sum_{(l)=(3)+1}^{(\overline{n})} \phi_{(l)} f_{(l)(3)} \right) \\
&\vdots \\
&\gamma_{(\overline{n})} = \phi_{(\overline{n})} \sum_{(l)=(1)}^{(\overline{n})-(1)} \phi_{(l)} f_{(\overline{n})(l)} > 0.
\end{aligned}
$$

Table 12: *Nonvanishing elements of* γ *under degeneracy and embedded spread operators under* $\overline{n} = 2$

(ii) Embedded spread pricing operators: Set $\overline{n} = 2$ in Table 12. (Equivalently, for the embedded measure (35), let $\phi_s = \phi_i$ precede ϕ_j. Then $\gamma_i = -\phi_i \sum_{l=i+1}^{n} \phi_l f_{li} = -\phi_i \phi_j f_{ji} < 0$. Let $\phi_s = \phi_j$ follow ϕ_i. Then $\gamma_j = \phi_j \sum_{l=1}^{j-1} \phi_l f_{jl} = \phi_j \phi_i f_{ji} > 0$.)

(iii) Natural ordering of spread pricing operators: Since $\gamma_F < 0 < \gamma_L$ for all $\phi \to \{\phi^{ji}\}$, spread pricing operators are naturally ordered irrespective of their spread measure.

A.10 Theorem 13: One-sided gradient restrictions

Under nontriviality of risk in respect of the $(j,i)'th$ spread, $rc = sc_{ji} = f_{ji}\alpha_{ji}\Delta x_{ji} > 0$ iff $\Delta x_{ji} > 0$ iff $\lambda \alpha_{ji} \Delta x_{ji} \Delta W_{ji} > 0$ iff $COV_{\phi \to \{\phi^{ji}\}}[x, W] > 0$. From (36) this condition is equivalent to $\nabla \phi^{ji} > \nabla \pi^{ji}$.

A.11 Theorem 14: Bounded shadow prices

A.11.1 Necessity

Because $Max\ \{f\} = f_{n1}$ under the natural ordering of the market portfolio, choose the embedded set of measures

$$\left\{\phi^{n1}\right\} \equiv \{\phi_1 \neq 0,\ 0, \ldots,\ 0,\ \phi_n \neq 0\}.$$

Hence applied to $\Sigma f = \phi - \pi$ Theorem 12 implies that $\phi_n - \pi_n = \phi_n \phi_1 f_{n1}$ so that $\pi_n = \phi_n (1 - \phi_1 f_{n1})$. π_n is an admissible risk neutral measure iff $\pi_n \in (0,1)$. Suppose $f_{n1} > 1$. Then there exist (ϕ_1, ϕ_n) pairs such that $\pi_n < 0$ implying an inadmissible risk neutral measure. Hence for arbitrary $\phi_1, \phi_n = 1 - \phi_1 \in (0,1)$ it is necessary that $0 < f_{n1} \leq 1$ for $0 < \pi_n = 1 - \pi_1 < 1$. Thus $0 < f_{n1} \leq 1$ is a necessary condition, whence $\mathbf{0} < f \leq \mathbf{1}$ is also necessary.

A.11.2 Sufficiency

Suppose $\mathbf{0} < f \leq \mathbf{1}$. For

$$\gamma_s = \phi_s \sum_{l=1}^{s-1} \phi_l f_{sl} - \phi_s \sum_{l=s+1}^{n} \phi_l f_{ls}, \tag{43}$$

the measure normalizations $\sum_{k \neq s} \phi_k^* \equiv \sum_{k \neq s} \phi_k / \overline{\phi_s} = 1$, where $\overline{\phi_s} \equiv 1 - \phi_s$, imply:

$$\gamma_s = \phi_s \overline{\phi_s} \sum_{l=1}^{s-1} \phi_l^* f_{sl} - \phi_s \overline{\phi_s} \sum_{l=s+1}^{n} \phi_l^* f_{ls}. \tag{44}$$

Subject to the hypothesis, successively maximise and minimise γ_s over all arbitrary nondegenerate $SPM = \phi$ to show that it is bounded inside the unit ball.

Observe that for the first summation in (43) its probabilities $\left\{\phi_1, \ldots, \phi_{s-1}\right\}$ precede ϕ_s, whereas for the second summation ϕ_s precedes $\left\{\phi_{s+1}, \ldots, \phi_n\right\}$. Hence

$$\left\{\left\{\phi_1, \ldots, \phi_{s-1}\right\}, \{\phi_s\}, \left\{\phi_{s+1}, \ldots, \phi_n\right\}\right\} \equiv \phi$$

partitions the elements of $SPM = \phi$ in natural order for all $s = 1, \ldots, n$ on the understanding that $\{\phi_1, \phi_0\} \equiv \varnothing$ and $\left\{\phi_{n+1}, \phi_n\right\} \equiv \varnothing$.

Maximisation For maximisation, first set $\left\{\phi_{s+1}, \ldots, \phi_n\right\} = 0$ to remove the negative term of (43). In respect of the surviving term of (44), from $\left\{\phi_1, \ldots, \phi_{s-1}\right\}$ set $\phi_1^* = 1$, or $\phi_1 = \overline{\phi_s}$. Then the specification

$$SPM = \left\{\phi_1 = \overline{\phi_s};\ \phi_2 = \cdots = \phi_{s-1} = 0;\ \phi_s > 0;\ \phi_{s+1} = \cdots = \phi_n = 0\right\}$$

for the subjective measure, satisfying the weak condition $\phi_i \neq 1$, selects the $(s,1)'th$ embedded measure and supplies the maximum

$$Max\left\{\gamma_s\right\} = Max\left\{\phi_s\overline{\phi_s}f_{s1}\right\} = \tfrac{1}{4}f_{s1},$$

since $f_{s1} = Max_l\left\{f_{sl}\right\}$ under the natural ordering. Hence under the hypothesis $\mathbf{0} < f \leq \mathbf{1}$,

$$0 < Max\left\{\gamma_s\right\} < 1, \qquad (s = 1, \ldots, n)\,. \tag{45}$$

Minimisation For minimisation, set $\left\{\phi_1, \ldots, \phi_{s-1}\right\} = 0$ to remove the positive term from (43). In respect of the surviving term of (44), from $\left\{\phi_{s+1}, \ldots, \phi_n\right\}$ set $\phi_n^* = 1$, or $\phi_n = \overline{\phi_s}$. Then the specification

$$SPM = \left\{\phi_1 = \cdots = \phi_{s-1} = 0;\ \phi_s > 0;\ \phi_{s+1} = \cdots = \phi_{n-1} = 0;\ \phi_n = \overline{\phi_s}\right\}$$

for the subjective measure, satisfying the nondegeneracy restriction $\phi_i \neq 1$, selects the $(n,s)'th$ embedded measure and supplies the minimum

$$Min\left\{\gamma_s\right\} = Min\left\{-\phi_s\overline{\phi_s}f_{ns}\right\} = -\tfrac{1}{4}f_{ns},$$

since $f_{ns} = Max_l\left\{f_{ls}\right\}$ under the natural ordering. Hence under the hypothesis $\mathbf{0} < f \leq \mathbf{1}$,

$$-1 < Min\left\{\gamma_s\right\} < 0, \qquad (s = 1, \ldots, n)\,. \tag{46}$$

Unit bounds imply existence of risk neutral measure Under the hypothesis $\mathbf{0} < f \leq \mathbf{1}$, (45) and (46) establish that $-1 < \gamma_s < 1$, for $s = 1, \ldots, n$, so that γ is a risk pricing operator bounded inside the unit ball over all nondegenerate choices ($\phi_i \neq 1$) of $SPM = \phi$ and also satisfying the one-sided restrictions required by Theorem 13. Hence $\pi = \phi - \gamma$ is an admissible risk neutral measure, for any such choice of $SPM = \phi$, so that Definition 11 is satisfied. Observe the crucial role of the embedded measures in both the necessity and sufficiency proofs. Here, for each choice of s, maximisation selects the $(s, 1)'th$ embedded measure and minimisation selects the $(n, s)'th$ embedded measure. The logic of that choice is clear: for each payoff s, always select the two payoffs 1 and n furthest away from it (only one payoff for the two boundary cases) to ensure, relative to payoff s and given the natural ordering on the shadow prices, maximisation and minimisation respectively.

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