On-line Appendix for "Debt, Policy Uncertainty and Expectations Stablization"

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Abstract

This appendix provides i) some calculations underlying the model; ii) some proofs and related detail not contained in the appendix of the paper; and iii) some additional results that both clarify our findings relative to earlier learning analyses on this topic and elucidate further the role of some assumptions.

Contents

1 Model Derivation

The following describes only the derivation of the aggregate demand equation. The derivation of the generalized Phillips curve can be found in Preston (2005b). The household's optimality conditions imply:

$$E_0^i \sum_{t=0}^{\infty} \beta^t \frac{U_c\left(C_t^i + g\right)}{U\left(C_0^i + g\right)} C_t^i = \frac{B_0^i}{P_0} + E_0^i \sum_{t=0}^{\infty} \beta^t \frac{U_c\left(C_t^i + g\right)}{U_c\left(C_0^i + g\right)} \left[Y_t - \frac{T_t}{P_t}\right]$$

which can be rewritten as

$$b_0^i \pi_0^{-1} = E_0^i \sum_{t=0}^{\infty} \beta^t \frac{U_c\left(C_t^i + g\right)}{U\left(C_0^i + g\right)} \left[C_t^i - Y_t + \tau_t\right] \tag{1}$$

where $\tau_t = T_t/P_t$ and $b_t^i = B_t^i/P_t$. In steady $\bar{s} = (1-\beta)\,\bar{b}$ where $s_t = T_t/P_t - g$ defines the structural surplus and market clearing implies $\bar{Y} = \bar{C} + g$.

Approximating (1) provides

$$\begin{aligned} \bar{b}\left(\hat{b}_0^i - \hat{\pi}_0\right) &= \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [\bar{C}\hat{C}_t^i - \bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t] + \bar{s}\hat{E}_0^i \sum_{t=0}^{\infty} \beta^t \left[\frac{U_{cc}}{U_c}\bar{C}\hat{C}_t^i - \frac{U_{cc}}{U_c}\bar{C}\hat{C}_0^i\right] \\ &= \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [\left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right)\hat{C}_t^i + \bar{s}\tilde{\sigma}^{-1}\hat{C}_0^i (1-\beta)^{-1} + \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [-\bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t] \end{aligned}$$

where $\tilde{\sigma} = -U_c/\left(U_{cc}\bar{C}\right)$.

The consumption Euler equation satisfies the log-linear approximation

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \tilde{\sigma}\left(\hat{\imath}_t - \hat{E}_t^i \hat{\pi}_{t+1}\right).$$

Solving recursively backwards and taking expectations at time zero provides

$$\hat{E}_0^i \hat{C}_t^i = \hat{C}_0^i + \tilde{\sigma}\hat{E}_0^i \sum_{s=0}^{t-1} \left(\hat{\imath}_s - \hat{\pi}_{s+1}\right).$$

This determines the infinite sum

$$\begin{aligned} \hat{E}_0^i \sum_{t=1}^{\infty} \beta^t \hat{C}_t^i &= \frac{\beta \hat{C}_0^i}{(1-\beta)} + \tilde{\sigma}\hat{E}_0^i \sum_{t=1}^{\infty} \beta^t \sum_{s=0}^{t-1} \left(i_s - \pi_{s+1}\right) \\ &= \frac{\beta \hat{C}_0^i}{(1-\beta)} + \frac{\tilde{\sigma}\beta}{(1-\beta)} \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t \left(i_t - \pi_{t+1}\right). \end{aligned}$$

Substituting into the intertemporal budget constraint

$$\begin{aligned} \bar{b}\left(\hat{b}_0^i - \hat{\pi}_0\right) &= \hat{E}_0^i \sum_{t=1}^{\infty} \beta^t \left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right)\hat{C}_t^i + \left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right)\hat{C}_0^i + \bar{s}\tilde{\sigma}^{-1}\frac{\hat{C}_0^i}{(1-\beta)} + \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [-\bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t] \\ &= \frac{\bar{C}\hat{C}_0}{(1-\beta)} + \left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right)\frac{\tilde{\sigma}\beta}{(1-\beta)} \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t \left(i_t - \pi_{t+1}\right) + \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [-\bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t] \end{aligned}$$

Divide through by $\bar{Y}\ (1-\beta)^{-1}$ and rearranging gives the optimal consumption rule

$$\hat{C}_0^i = s_C^{-1}\delta\left(\hat{b}_0 - \hat{\pi}_0\right) + s_C^{-1}\hat{E}_0^i\sum_{t=0}^{\infty}\beta^t\left[(1-\beta)\left(\hat{Y}_t - \delta\hat{s}_t\right) - (\sigma-\delta)\,\beta\,(\hat{\imath}_t - \hat{\pi}_{t+1})\right]$$

where $s_C = \bar{C}/\bar{Y}$, $\delta = \bar{s}/\bar{Y}$, $\bar{s}\hat{s}_t = \bar{\tau}\hat{\tau}$, $\sigma = s_c\tilde{\sigma}$ and using $\bar{s} = (1-\beta)\,\bar{b}$.

Finally, note that market clearing implies the log-linear approximations

$$\hat{Y}_t = s_C\int_0^1 \hat{C}_t^i di \text{ and } \hat{b}_t = \int_0^1 \hat{b}_t^i di.$$

Hence the aggregate demand equation is

$$\hat{Y}_0 = \delta\left(\hat{b}_0 - \hat{\pi}_0\right) + \hat{E}_0\sum_{t=0}^{\infty}\beta^t\left[(1-\beta)\left(\hat{Y}_t - \delta\hat{s}_t\right) - (\sigma-\delta)\,\beta\,(\hat{\imath}_t - \hat{\pi}_{t+1})\right]$$

where $\hat{E}_t = \int_0^1 \hat{E}_t^i di$ defines average beliefs of households.

Define the output gap as $Y_t - Y_t^n$ where the latter is the natural rate of output under rational expectations permits

$$x_t = \delta\beta^{-1}\left(\hat{b}_t - \hat{\pi}_t\right) + \hat{E}_t\sum_{T=t}^{\infty}\beta^t\left[(1-\beta)\left(\hat{x}_{T+1} - \delta\beta^{-1}\hat{s}_T\right) - (\sigma-\delta)\,(\hat{\imath}_t - \hat{\pi}_{t+1}) + r_T\right]$$

where

$$r_t = Y_{t+1}^n - Y_t^n.$$

Equation (10) of the paper follows when $\tilde{\sigma} = 1$ and assuming $g = 0$, without loss of generality, so that $\tilde{\sigma} = \sigma = 1$.

2 Proof of Proposition 1

The model under rational expectations is given by

$$\begin{aligned}
\hat{x}_t &= E_t\hat{x}_{t+1} - (\hat{\imath}_t - E_t\hat{\pi}_{t+1} - r_t^n) \\
\hat{\pi}_t &= \kappa\hat{x}_t + \beta E_t\hat{\pi}_{t+1.}
\end{aligned}$$

The debt dynamics and policy rules are as specified earlier. Solving the model under Ricardian fiscal policy is standard and yields $\hat{\pi}_t = \phi_0 r_t$ where ϕ_0 is a time-invariant coefficient, the specific value of which plays not role for the stability results under both learning and rational expectations. To solve the model under the assumption of non-Ricardian fiscal policy note that the governments flow budget constraint is again solved forward to give

$$\begin{aligned}\hat{b}_t &= \frac{\beta}{1-(1-\beta)\,\phi_\tau}\hat{b}_{t+1} + \frac{(1-\beta\phi_\pi)}{1-(1-\beta)\,\phi_\tau}\hat{\pi}_t \\ &= \frac{(1-\beta\phi_\pi)}{1-(1-\beta)\,\phi_\tau}\sum_{j=0}^{\infty}\left(\frac{\beta}{1-(1-\beta)\,\phi_\tau}\right)^j E_t\hat{\pi}_{t+j}. \end{aligned} \tag{2}$$

Since the eigenvalue pertaining to debt is great than one that implies that only one of the remaining two eigenvalues can lie in the unit circle.

Considering the sub-system in output and prices gives

$$E_t\begin{bmatrix}\hat{\pi}_{t+1}\\ \hat{x}_{t+1}\end{bmatrix} = \begin{bmatrix}\beta^{-1} & -\kappa\beta^{-1}\\ \left(\phi_\pi-\beta^{-1}\right) & 1+\kappa\beta^{-1}\end{bmatrix}\begin{bmatrix}\hat{\pi}_t\\ \hat{x}_t\end{bmatrix} - \begin{bmatrix}0\\ \sigma\end{bmatrix} r_t^n$$

with associated characteristic equation

$$P(\lambda) = \lambda^2 - \left(\frac{1+\beta+\kappa}{\beta}\right)\lambda + \left(\frac{1+\kappa\phi_\pi}{\beta}\right).$$

There will be one eigenvalue inside the unit circle if and only if $\phi_\pi < 1$.

Denoting the unstable root by λ_1, the associated eigenvector can be determined from

$$\begin{bmatrix}1 & d_1\end{bmatrix}\begin{bmatrix}\beta^{-1} & -\kappa\beta^{-1}\\ \left(\phi_\pi-\beta^{-1}\right) & 1+\kappa\beta^{-1}\end{bmatrix} = \begin{bmatrix}1 & d_1\end{bmatrix}\lambda_1$$

implying that

$$\begin{aligned}d_1 &= \frac{\kappa\beta^{-1}}{1+\kappa\beta^{-1}-\lambda_1}\\ &= \frac{\kappa}{\beta\lambda_2-1}\end{aligned}$$

where the second equality follows from noting that the roots satisfy $\lambda_1+\lambda_2 = (1+\beta+\kappa)\,\beta^{-1}$. Transforming the system with the unstable eigenvector gives

$$E_t\bar{z}_{t+1} = \lambda_1\bar{z}_t + \frac{\kappa}{1-\lambda_2\beta}r_t^n$$

where

$$\bar{z}_t = \begin{bmatrix}1 & d_1\end{bmatrix}\begin{bmatrix}\hat{\pi}_t\\ \hat{x}_t\end{bmatrix}.$$

Solving forward provides

$$\hat{\pi}_t = \frac{\kappa}{1-\beta\lambda_2}\hat{x}_t + \frac{\kappa}{(\beta\lambda_2-1)\,\lambda_1}r_t^n$$

placing a linear restriction on output and inflation movements in equilibrium.

Substitution into the Phillips curve gives

$$E_t\hat{\pi}_{t+1} = \lambda_2\hat{\pi}_t - \frac{\beta^{-1}}{\lambda_1}r_t^n$$

and solving backwards recursively and taking expectations at time t implies

$$E_t\hat{\pi}_{t+j} = \lambda_2^j\hat{\pi}_t - \frac{\beta^{-1}}{\lambda_1}\lambda_2^{j-1}r_t^n.$$

Using this to evaluate the expectations in the debt equations gives

$$\begin{aligned} \hat{\pi}_t &= \tilde{\phi}_1^{-1}\hat{b}_t + \tilde{\phi}_1^{-1}\tilde{\phi}_2 r_t && (3)\\ &= \phi_1\hat{b}_t + \phi_2 r_t && (4) \end{aligned}$$

where

$$\begin{aligned} \tilde{\phi}_1 &= \frac{(1-\beta\phi_\pi)}{1-(1-\beta)\,\phi_\tau - \beta\lambda_2} \\ \tilde{\phi}_2 &= \frac{(1-\beta\phi_\pi)}{\lambda_1\,(1-(1-\beta)\,\phi_\tau - \beta\lambda_2)\,(1-(1-\beta)\,\phi_\tau)} \end{aligned}$$

and

$$\lambda_2 = \frac{1}{2\beta}\left[1+\beta+\kappa-\sqrt{(1+\beta+\kappa)^2-4\beta(1+\kappa\phi_\pi)}\right].$$

3 Proofs of Propositions with Learning Dynamics

3.1 Constructing the True Data Generating Process

This section outlines the beliefs of agents in our benchmark analysis. We re-write the model in matrix form. Each agent's estimated model at date t can be expressed as

$$X_t = \begin{bmatrix} x_t & \pi_t & b_{t+1} & i_t & s_t \end{bmatrix}' = \omega_{0,t} + \omega_{1,t}X_{t-1} + \bar{e}_t \tag{5}$$

where ω_0 denotes the constant, ω_1 is defined as

$$\omega_1 = \begin{bmatrix} 0 & b_x^\pi & b_x^b & 0 & 0 \\ 0 & b_\pi^\pi & b_\pi^b & 0 & 0 \\ 0 & b_b^\pi & b_b^b & 0 & 0 \\ 0 & b_i^\pi & b_i^b & 0 & 0 \\ 0 & b_s^\pi & b_s^b & 0 & 0 \end{bmatrix}.$$

and $\bar{e}_t$ represents an i.i.d. estimation error.

The model, given by equations (6), (7), (8), (10) and (11) of the paper, can be written as:

Output gap

$$\Psi_x^1 X_t = \Psi_x^2 \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} X_{T+1} + r_t^n$$

where

$$\Psi_x^1 = \begin{bmatrix} 1 & 0 & 1 & 0 & -\delta \end{bmatrix} \text{ and } \Psi_x^2 = \begin{bmatrix} 1-\beta & 1-\delta & (1-\delta)\beta & (1-\beta)\delta & 0 \end{bmatrix}.$$

Inflation

$$\Psi_\pi^1 X_t = \Psi_\pi^2 \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} X_{T+1}$$

where

$$\Psi_\pi^1 = \begin{bmatrix} -\kappa & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \Psi_\pi^2 = \begin{bmatrix} \kappa\alpha\beta & (1-\alpha)\beta & 0 & 0 & 0 \end{bmatrix}.$$

Interest rate

$$\Psi_i^1 X_t = \Psi_i^2 \omega_0 + \Psi_i^2 \omega_1 X_{t-1}$$

where

$$\Psi_i^1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } \Psi_i^2 = \begin{bmatrix} 0 & \phi_\pi & 0 & 0 & 0 \end{bmatrix}.$$

Surplus

$$\Psi_s^1 X_t = \Psi_s^2 X_{t-1}$$

where

$$\Psi_s^1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \Psi_s^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \phi_\tau \end{bmatrix}.$$

Debt

$$\Psi_b^1 X_t = \Psi_b^2 X_{t-1}$$

where

$$\Psi_b^1 = \begin{bmatrix} 0 & \beta^{-1} & -1 & \beta^{-1}(1-\beta) & 1 \end{bmatrix} \text{ and } \Psi_b^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \beta^{-1} \end{bmatrix}.$$

Calculating expectations over an infinite horizon provides

$$\begin{aligned} \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} X_{T+1} &= (I-\omega_1)^{-1} \left(I \cdot (1-\beta)^{-1} - \omega_1 (I - \beta\omega_1)^{-1} \right) \omega_0 \\ &\quad + \omega_1 (I - \beta\omega_1)^{-1} \\ &= F_{x0}(\omega_0, \omega_1) + F_{x1}(\omega_1) X_{t-1} \end{aligned}$$

and

$$
\begin{aligned}
\hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} X_{T+1} &= (I-\omega_1)^{-1} \left(I \cdot (1-\alpha\beta)^{-1} - \omega_1 (I-\beta\omega_1)^{-1} \right) \omega_0 \\
&\quad + \omega_1 (I - \alpha\beta\omega_1.)^{-1} X_{t-1} \\
&= F_{\pi 0}(\omega_0, \omega_1) + F_{\pi 1}(\omega_1) X_{t-1}.
\end{aligned}
$$

The true data generating process is then

$$
X_t = [A_0(\omega_1)]^{-1} [A_1(\omega_0, \omega_1) + A_2(\omega_1)] X_{t-1} + [A_0(\omega_1)]^{-1} r_t^n
$$

where

$$
A_0(\omega_1) = \begin{bmatrix} \Psi_x^1 - \Psi_x^2 F_{x1}(\omega_1) \\ \Psi_\pi^1 - \Psi_\pi^2 F_{\pi 1}(\omega_1) \\ \Psi_i^1 \\ \Psi_s^1 \\ \Psi_b^1 \end{bmatrix}
$$

and

$$
A_1(\omega_0, \omega_1) = \begin{bmatrix} \Psi_x^2 F_{x0}(\omega_0, \omega_1) \\ \Psi_\pi^2 F_{\pi 0}(\omega_0, \omega_1) \\ \Psi_i^2 \omega_0 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \text{ and } A_2(\omega_1) = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \Psi_i^2 \omega_1 \\ \Psi_s^2 \\ \Psi_b^2 \end{bmatrix}.
$$

Finally, the data generating process can be rearranged as

$$
\begin{aligned}
X_t &= A_0(\omega_1)^{-1} A_1(\omega_0, \omega_1) + A_0(\omega_1)^{-1} A_2(\omega_1) X_{t-1} + A_0(\omega_1)^{-1} r_t^n \\
&= \Gamma_1(\omega_0, \omega_1) \begin{bmatrix} 1 \\ X_{t-1} \end{bmatrix},
\end{aligned}
$$

E-stability can be computed by evaluating the local stability of the following ODE

$$
\frac{d(\omega_0, \omega_1)}{d\tau} = \Gamma_1(\omega_0, \omega_1) - (\omega_0, \omega_1). \tag{6}
$$

3.2 Stability under learning

As explained above, convergence of the learning process depends on the stability properties of the ODE (6). This is a fairly complicated convolutions of the agent's beliefs (ω_0, ω_1). In order to find analytical conditions for convergence we make use of Matlab symbolic toolbox. The expressions can be reproduced by running the appropriate files, available on request from the authors.

3.3 Proof of Proposition 2

Ricardian regime The results reported in this proof can be reproduced using the Matlab file: *fiscal_ delay_ benchmark.m.* First, it can be shown that the beliefs (ω_0, ω_1) evolve according to two separate sub-systems

$$\dot{\omega}_0 = (A - I_5)\,\omega_0 \text{ and } \dot{\omega}_1 = (B - I_5)\,\omega_1.$$

where A and B represent components of the Jacobian of $\bar{T}\,(\omega_0, \omega_1)$, evaluated at the rational expectations equilibrium ω_0^*, ω_1^*, defined in the previous proposition.

Consider the evolution of the intercept ω_0. We take three steps in order to reduce the matrix A from a five dimensional to three dimensional object. First, evaluating the matrix A reveals that

$$\dot{\omega}_0^s = -\omega_0^s.$$

Hence, the intercept in the fiscal rule equation converges for all parameter values, independently of the other elements of the beliefs vector. This reduces dimensionality by one. Second, using the restriction

$$A_{5,j} = -\beta^{-1} A_{2,j} + A_{3,j} \quad \text{for } j = 1...5$$

delivers the three dimension system (see the Matlab file for the details of the variables' transformation):

$$\begin{bmatrix} \dot{\omega}_0^x \\ \dot{\omega}_0^\pi \\ \dot{\omega}_0^i \end{bmatrix} = \tilde{A} \begin{bmatrix} \omega_0^x \\ \omega_0^\pi \\ \omega_0^i \end{bmatrix}.$$

For the real parts of the three eigenvalues to be negative requires

$$Tr(\tilde{A}) < 0, \quad \det(\tilde{A}) < 0$$

and

$$M_{\tilde{A}} = -Sm(\tilde{A}) \cdot Tr(\tilde{A}) + \det(\tilde{A}) > 0$$

where $Sm(\tilde{A})$ denotes the sum of all principle minors of $\tilde{A}$. We are interested in the limit case where $\alpha \to 0$. In this case, the trace, determinant and $M_{\tilde{A}}$ become arbitrarily large. Consider the trace first. We can calculate the limit

$$\lim_{\alpha \to 0^+} \alpha \cdot Tr(\tilde{A}) = -\frac{\phi_\pi - 1 - \phi_\pi \beta}{1 - \beta}$$

which is negative if and only if

$$\phi_\pi > \frac{1}{1-\beta}. \tag{7}$$

Likewise, the determinant

$$\lim_{\alpha\to 0^+} \alpha \det(\tilde{A}) = -\frac{\phi_\pi - 1}{1-\beta}$$

which is negative if and only if $\phi_\pi > 1$. For $M_{\tilde{A}}$ we have

$$\lim_{\alpha\to 0^+} \alpha^2 \cdot M_{\tilde{A}} = \frac{(2 - 2\phi_\pi + \phi_\pi\beta)(1 - \phi_\pi + \phi_\pi\beta)}{(1-\beta)^2}$$

which is positive provided (7).

Consider now the coefficients on real debt. An identical process reduces the dimensionality of the matrix B to a three dimensional matrix $\tilde{B}$. Considering the trace we get

$$\lim_{\alpha\to 0^+} \alpha \cdot Tr(\tilde{B}) = \frac{1 - \phi_\tau (1-\beta)(\beta\phi_\pi + 1)}{\phi_\tau \beta (1-\beta)}$$

which is decreasing in ϕ_τ. In a Ricardian regime, $\phi_\tau > 1$. Evaluating the expression at $\phi_\tau = 1$, if (7) then the trace of the $\tilde{B}$ matrix is negative. Evaluating the determinant we get

$$\lim_{\alpha\to 0^+} \alpha \cdot \det(\tilde{B}) = \frac{1 - \beta\phi_\pi - \beta^{-1}\phi_\tau\beta(1-\beta)}{\phi_\tau\beta(1-\beta)}$$

which is decreasing in ϕ_τ. Again, imposing $\phi_\tau = 1$ gives

$$\lim_{\alpha\to 0^+} \alpha \cdot \det(\tilde{B}) = -\frac{\phi_\pi - 1}{\phi_\tau(1-\beta)} < 0.$$

Finally,

$$\lim_{\alpha\to 0^+} \alpha^2 \cdot M_{\tilde{B}} = \frac{[\phi_\tau (\beta - 1)(\beta\phi_\pi + 2) - \beta\phi_\pi + 2]\,[\phi_\tau (\beta - 1)(\beta\phi_\pi + 1) + 1]}{\beta^2\phi_\tau^2(1-\beta)^2}.$$

which is, again, decreasing in ϕ_τ.Imposing $\phi_\tau = 1$ yields

$$\lim_{\alpha\to 0^+} \alpha^2 \cdot M_{\tilde{B}} = \frac{(2 - 2\phi_\pi + \beta\phi_\pi)(1 - \phi_\pi + \beta\phi_\pi)}{\beta\phi_\tau^2(1-\beta)^2}$$

which is positive if (7) is satisfied.

Non-Ricardian regime The matrices A and B corresponding to the non-Ricardian regime can be reduced to three dimensional matrices by following the same steps as above. To further simplify the problem, we use two Lemmas.

Lemma 1 *Consider the model where* $\alpha \to 0$. *Then* $\lambda_2 \to \phi_\pi$.

Proof. Recall that

$$\lambda_2 = \frac{1}{2\beta}\left[1+\beta+\kappa\left(\alpha\right)-\sqrt{(1+\beta+\kappa\left(\alpha\right))^2-4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}\right].$$

We can then evaluate

$$\lim_{\alpha\to 0}\lambda_2 =$$

$$\frac{1}{2\beta}\lim_{\alpha\to 0}\frac{\left\{\begin{array}{c}\left[1+\beta+\kappa\left(\alpha\right)+\sqrt{(1+\beta+\kappa\left(\alpha\right))^2-4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}\right]\times\\ \left[1+\beta+\kappa\left(\alpha\right)-\sqrt{(1+\beta+\kappa\left(\alpha\right))^2-4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}\right]\end{array}\right\}}{\left[1+\beta+\kappa\left(\alpha\right)+\sqrt{(1+\beta+\kappa\left(\alpha\right))^2-4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}\right]} =$$

$$\frac{1}{2\beta}\lim_{\alpha\to 0}\frac{4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}{\left[1+\beta+\kappa\left(\alpha\right)+\sqrt{(1+\beta+\kappa\left(\alpha\right))^2-4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}\right]}.$$

Using L'Hôpital

$$\frac{1}{2\beta}\lim_{\alpha\to 0}\left[4\beta\kappa'\left(\alpha\right)\phi_\pi/\left(\kappa'\left(\alpha\right)+\frac{(1+\beta+\kappa\left(\alpha\right))\kappa'\left(\alpha\right)-2\beta\kappa'\left(\alpha\right)\phi_\pi}{\sqrt{(1+\beta+\kappa\left(\alpha\right))^2-4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}}\right)\right] =$$

$$\begin{aligned}\frac{1}{2\beta}\lim_{\alpha\to 0}\left[4\beta\phi_\pi/\left(1+\frac{(1+\beta+\kappa\left(\alpha\right))-2\beta\phi_\pi}{\sqrt{(1+\beta+\kappa\left(\alpha\right))^2-4\beta(1+\kappa\left(\alpha\right)\phi_\pi)}}\right)\right] &= \\ \frac{1}{2\beta}\lim_{\alpha\to 0}2\beta\phi_\pi &= \phi_\pi.\end{aligned}$$

■

We then conjecture that as $\alpha\to 0$, one eigenvalue of $\tilde{A}$ and $\tilde{B}$ tends to -1. The conjecture is verified in the following Lemma.

Lemma 2 *Consider the model where $\alpha\to 0$. Then one eigenvalue ψ of $\tilde{A}$ and $\tilde{B}$ converges to -1.*

Proof. The characteristic equations of $\tilde{A}$ and $\tilde{B}$ are

$$\Delta^{\tilde{A}}\left(\psi\right)=\psi^3-tr\left(\tilde{A}\right)\psi^2+Sm(\tilde{A})\psi-\det(\tilde{A})$$

and

$$\Delta^{\tilde{B}}\left(\psi\right)=\psi^3-tr\left(\tilde{B}\right)\psi^2+Sm(\tilde{B})\psi-\det(\tilde{B}).$$

It can be shown that[1]

$$\lim_{\alpha\to 0}\Delta^{\tilde{A}}\left(-1\right)=-1-tr\left(\tilde{A}\right)-Sm(\tilde{A})-\det(\tilde{A})=0$$

[1] The limit is computed in the matlab file fiscal_delay_benchmark.m.

and

$$\lim_{\alpha\to 0}\Delta^{\tilde{B}}(-1) = -1 - tr\left(\tilde{B}\right) - Sm(\tilde{B}) - \det(\tilde{B}) = 0.$$

■

Let us consider the local stability of the intercept coefficients. The remaining two eigenvalues z_1 and z_2 of $\tilde{A}$ are negative if

$$tr(\tilde{A}) + 1 = z_1 + z_2 < 0 \text{ and } \det(\tilde{A}) = -z_1 z_2 < 0.$$

The trace is

$$tr\left(\tilde{A}\right) = -\left[1 + \frac{1 - [1 - \beta\phi_\pi(1-\beta)]\,\phi_\tau - \beta\phi_\pi}{1 - (1-\beta)\phi_\tau - \beta\phi_\pi}\right]. \tag{8}$$

(a) Consider the case of $0 \le \phi_\tau < 1$: the trace can be re-arranged to deliver the following relationship between ϕ_τ and ϕ_π at $tr\left(\tilde{A}\right) = 0$,

$$\phi_\tau = \frac{2}{[(1-\beta\phi_\pi)^{-1} + (1-\beta)]}$$

in the text. Then, $0 \le \phi_\tau < \min\left(\phi_\tau^*\left(\phi_\pi\right), 1\right)$, where

$$\phi_\tau^*\left(\phi_\pi\right) = \frac{2}{[(1-\beta\phi_\pi)^{-1} + (1-\beta)]},$$

and where we use $\partial tr\left(A\right)/\partial\phi_\tau > 0$ for $\phi_\tau \in [0, 1)$. The determinant is

$$-\det\left(\tilde{A}\right) = \frac{\left(1-\phi_\tau\right)\left(\beta\phi_\pi - 1\right)}{\left(1-\beta\right)\phi_\tau + \beta\phi_\pi - 1} > 0. \tag{9}$$

Finally, consider the B matrix. Proceeding in the same way as for the $\tilde{A}$ matrix, the trace can be shown to be

$$tr(\tilde{B}) = -2 - \frac{(1-\beta)\beta^2\phi_\pi^2\phi_\tau}{(-(1-\beta)\phi_\tau - \beta\phi_\pi + 1)(\beta\phi_\pi - 1)}. \tag{10}$$

which gives the following expression for

$$\phi_\tau^{**}\left(\phi_\pi\right) = \frac{2}{\left(\beta^2\phi_\pi^2 + 2\left(1-\beta\phi_\pi\right)\right)\frac{(1-\beta)}{(1-\beta\phi_\pi)^2}}$$

which solves $tr(\tilde{B}) = 0$ (also shown to have positive derivative with respect to ϕ_τ). It can be shown that $\phi_\tau^{**}\left(\phi_\pi\right) > \phi_\tau^*\left(\phi_\pi\right)$.[2] The determinant of the $\tilde{B}$ matrix is equal to -1 for every parameter value.

(b) Straightforward algebraic manipulations of (8)-(10) show that the stability condition holds for all parameter values with $\phi_\tau > \left(1+\beta\right)/(1-\beta)$.

[2] It can be shown that the difference betwen the denominator of τ^* and the denominator in τ^{**} is equal to $\left(\beta\phi_\pi - 1\right)^{-2}\left(1-\phi_\pi\right)\beta > 0$.

3.4 Proof of Proposition 3

The proof follows the same steps as in Proposition 2.

Ricardian Regime. The matrices A and B are three dimensional, given that agents do not have to forecast the nominal interest rate and the surplus. The expressions below are calculated using the file *fiscal_ analytical_ trsp.m*. Let us consider first the matrix A. We find

$$\lim_{\alpha \to 0^+} \alpha \cdot tr(A) = \frac{1 + (\beta\delta - 1)\phi_\pi}{1 - \beta} \tag{11}$$

which gives the stability condition in the main text. Thus, for $\delta = 0$, we the Taylor principle obtains. Using $\delta = (1-\beta)\frac{\bar{b}}{\bar{y}}$ we can re-write the stability condition as

$$\phi_\pi(1 - \beta(1-\beta)\frac{\bar{b}}{\bar{y}}) - 1 > 0$$

so that for high levels of debt-to-output ratio and for intermediate values of the discount factor instability is likely to arise. As $\alpha \to 0$, the determinant is

$$\lim_{\alpha \to 0^+} \alpha \cdot \det(A) = -\frac{\phi_\pi - 1}{1 - \beta}$$

and negative provided $(\phi_\pi - 1) > 1$. Finally,

$$\lim_{\alpha \to 0^+} \alpha^2 \left[-Sm(A) \cdot Tr(A) + \det(A)\right] = \frac{\left(\phi_\pi \left(2 - \beta\delta\right) - 2\right)\left(\phi_\pi \left(1 - \beta\delta\right) - 1\right)}{\left(1 - \beta\right)^2},$$

which is positive provided (11) is satisfied.

Consider now the matrix B. The trace is satisfies

$$\lim_{\alpha \to 0^+} \alpha \cdot tr(B) = \frac{(-1 + \beta + \beta^2\phi_\pi\delta - \beta\phi_\pi\delta)\phi_\tau - (1-\delta)\,\beta\phi_\pi + 1}{(1-\beta)\,\beta\phi_\tau}$$

and is negative provided the trace of the matrix for the constants is negative ($\phi_\tau > 1$ in the Ricardian fiscal regime). As $\alpha \to 0$, the determinant is always negative, that is

$$\lim_{\alpha \to 0^+} \alpha \cdot \det(B) = -\beta^{-1} - \frac{(\beta\phi_\pi - 1)}{(1-\beta)\,\beta\phi_\tau} < 0,$$

if (11) is satisfied.

Finally, letting $\alpha \to 0$, the sum of all principle minors becomes

$$\lim_{\alpha \to 0^+} \alpha^2 \left[-Sm(B) * Tr(B) + \det(B)\right] =$$

$$\frac{\left[(-2\beta - \beta^2\phi_\pi\delta + \beta\phi_\pi\delta + 2)\phi_\tau - 2 - \beta\phi_\pi\delta + 2\beta\phi_\pi\right]\left[(1 + \beta\phi_\pi\delta - \beta - \beta^2\phi_\pi\delta)\phi_\tau - 1 + \beta\phi_\pi - \beta\phi_\pi\delta\right]}{\phi_\tau^2\beta^2(1-\beta)^2,}$$

which is positive provided $\phi_\tau > 1$ (Ricardian fiscal Regime) and (11) is satisfied.

Non-Ricardian Regime. As in proposition 2, it can be shown that one eigenvalue of both matrices A and B is equal to -1. We define the trace of the constant coefficients as

$$\lim_{\alpha \to 0^+} tr(A) + 1 = \Phi^A(\phi_\tau, \phi_\pi, \delta).$$

First notice that

$$\Phi_\delta^A(\phi_\tau, \phi_\pi, \delta) = \frac{(1-\beta\phi_\pi)\beta^2\phi_\pi}{(1-\beta)\left(1-(1-\beta)\phi_\tau - \beta\phi_\pi\right)} > 0$$

if $0 \leq \phi_\tau < 1$ and

$$\Phi_{\phi_\tau}^A(\phi_\tau, \phi_\pi, \delta) = \frac{(1-\beta\phi_\pi)(\beta\phi_\pi\delta - \phi_\pi + 1)\beta}{(-1+\phi_\tau - \phi_\tau\beta + \beta\phi_\pi)^2} > 0$$

for all admissible values of δ, ϕ_π, and ϕ_τ, where Φ_x^A denotes the derivative of Φ^A with respect to the argument x. Second we show that for values of $\delta < \delta^{TA}$ the trace is negative. Consider $\phi_\tau < 1$. Using the inequality above, we can solve for δ^{TA} as

$$\Phi^A(1, \phi_\pi, \delta^{TA}) = -\frac{(\beta^2\phi_\pi - \beta^2\phi_\pi^2 + \beta^2\phi_\pi^2\delta^{TA} + \beta\phi_\pi - \beta\phi_\pi\delta^{TA} - \beta - \phi_\pi + 1)}{(1-\beta)(1-\phi_\pi)} = 0$$

where

$$\delta^{TA} = \frac{\left((1-\beta+\phi_\pi\beta^2\right)(1-\phi_\pi)}{\phi_\pi\beta(1-\beta\phi_\pi)} > 0.$$

If $\phi_\tau > (1+\beta)/(1-\beta)$ then $\Phi_\delta^A(\phi_\tau, \phi_\pi, \delta) < 0$. Evaluating Φ^A at $\delta = 0$ gives

$$\Phi^A(\phi_\tau, \phi_\pi, 0) = \frac{(C_{\phi_\tau}\phi_\tau - \beta^3\phi_\pi^2 + 3\beta^2\phi_\pi - 2\beta\phi_\pi - 2\beta + 2)}{(1-\beta)((1-\beta)\phi_\tau - 1 + \beta\phi_\pi)}$$

where

$$C_{\phi_\tau} = (\beta^3\phi_\pi - 2\beta^2\phi_\pi - \beta^2 + \beta\phi_\pi + 3\beta - 2).$$

The denominator is positive for $\phi_\tau > (1+\beta)/(1-\beta)$. For the numerator, substituting $\phi_\tau = ((1+\beta)/(1-\beta))$ gives

$$(C_{\phi_\tau}\phi_\tau - \beta^3\phi_\pi^2 + 3\beta^2\phi_\pi - 2\beta\phi_\pi - 2\beta + 2) = \left(\beta^2 - \beta\right) + \left(\beta^2\phi_\pi - \beta\phi_\pi\right) + (2\beta^2\phi_\pi - 2\beta) - \beta^3\phi_\pi - \beta^3\phi_\pi^2 < 0$$

Last, the coefficient C_{ϕ_τ} is positive since

$$\left(\beta^3\phi_\pi - \beta^2\phi_\pi\right) + R(\phi_\pi, \beta) < 0$$

where

$$R(\phi_\pi, \beta) = -\beta^2\phi_\pi - \beta^2 + \beta\phi_\pi + 3\beta - 2$$

$$R(0, \beta) = -(\beta-1)^2 - 1 + \beta < 0, \qquad R(1, \beta) = -2(\beta-1)^2 < 0$$

and

$$R_{\phi_\pi}(\phi_\pi, \beta) = -\beta^2 + \beta > 0.$$

Hence, for $\phi_\tau > (1+\beta)/(1-\beta)$ the trace is negative. Finally, the determinant of the Jacobian is

$$\lim_{\alpha \to 0^+} [-\det(A)] = \frac{(1-\phi_\tau)(1-\beta\phi_\pi)}{1-(1-\beta)\,\phi_\tau - \beta\phi_\pi} > 0.$$

Following the same steps for matrix B :

$$\lim_{\alpha \to 0^+} tr(B) + 1 = \Phi^B(\phi_\tau, \phi_\pi, \delta).$$

First notice that

$$\Phi^B_\delta(\phi_\tau, \phi_\pi, \delta) = \frac{\beta^2\phi_\pi^2}{(1-\beta)\,(1-(1-\beta)\,\phi_\tau - \beta\phi_\pi)} > 0$$

for $0 < \phi_\tau < 1$ and

$$\Phi^B_{\phi_\tau}(\phi_\tau, \phi_\pi, \delta) = \frac{(1-\beta)\delta\beta^2\phi_\pi^2}{(-1+\phi_\tau - \phi_\tau\beta + \beta\phi_\pi)^2} > 0.$$

Solving for δ^{TB} from

$$\Phi^B(1, \phi_\pi, \delta^{TB}) = \frac{(\beta^2\phi_\pi^3 - \beta^2\phi_\pi^3\delta^{TB} - \beta^2\phi_\pi^2 - 2\beta\phi_\pi^2 + \beta\phi_\pi^2\delta^{TB} + 2\beta\phi_\pi + 2\phi_\pi - 2)}{(1-\beta\phi_\pi)(1-\phi_\pi)} = 0$$

provides

$$\delta^{TB} = \frac{\left((1-\beta\phi_\pi) + \phi_\pi\beta^2 + (1-\beta\phi_\pi) - \phi_\pi\beta^2(1-\phi_\pi)\right)(1-\phi_\pi)}{\beta\phi_\pi^2\,(1-\beta\phi_\pi)} > \delta^{TA}.$$

Moreover, for $\phi_\tau > (1+\beta)/(1-\beta)$

$$tr(B) = \Phi^B_\delta(\phi_\tau, \phi_\pi, \delta) < 0, \quad \text{and} \quad \Phi^B(\phi_\tau, \phi_\pi, 0) = -(\beta^2\phi_\pi^2 - 2\beta\phi_\pi + 2)/(1-\beta\phi_\pi) < 0.$$

Finally, the determinant is equal to one for all parameter values.

3.5 Proof of Proposition 5

Ricardian regime. Given the results in the Propositions above, it is sufficient to evaluate the matrix A. Evaluating the expression (11) at $\delta = 1$ gives

$$1 + (\beta - 1)\phi_\pi < 0$$

which coincides with the stability condition in the case where the agents have no knowledge about the policy rule. Thus, communication is always stability enhancing. The case of $\delta = 0$ is trivial.

Non-Ricardian regime. Let $\delta = 1$. then

$$\begin{aligned}\Phi^A(\phi_\tau, \phi_\pi, 1) &= \frac{(\beta^2\phi_\pi\phi_\tau + 2\beta\phi_\pi - \beta\phi_\pi\phi_\tau - \phi_\tau\beta + 2\phi_\tau - 2)}{1-(1-\beta)\phi_\tau - \beta\phi_\pi} \\ &= -\left[1 + \frac{1 - [1-\beta\phi_\pi(1-\beta)]\,\phi_\tau - \beta\phi_\pi}{1-(1-\beta)\phi_\tau - \beta\phi_\pi}\right]\end{aligned}$$

which is the trace obtained about for the case of *no communication*. The function Φ^A is defined in the proof[3] of Proposition 4. Since $\Phi^A_\delta > 0$ for $0 < \phi_\tau < 1$ we have that $\Phi^A < \Phi^A(\phi_\tau, \phi_\pi, 1)$ for $\delta < 1$. Hence, communication improves stability. Finally,

$$\Phi^A(1, \phi_\pi, 0) = -\frac{\beta^2\phi_\pi - \beta^2\phi_\pi^2 + (1-\beta)(1-\phi_\pi)}{(1-\beta)(1-\phi_\pi)} < 0$$

thus restoring the Leeper conditions. The case where $\phi_\tau > (1+\beta)/(1-\beta)$ is trivial from the proof of proposition 4.

3.6 Proof of Proposition 6

As in proposition 3, we can show that

$$\lim_{\alpha\to 0} \alpha \cdot tr\,(A) = \frac{\sigma + \delta\phi_\pi\beta - \phi_\pi\sigma}{(1-\beta)\,\sigma}$$

which gives the desired result. The expressions below are calculated using the file *fiscal_ analytical_ trsp.m.*

3.7 Proof of Proposition 7

Take the conditions of proposition 3 and set $\delta = 0$ to obtain the result — the zero debt economy being isomorphic to an economy in which fiscal expectations are anchored.

4 Additional Propositions

Two further results are reference without proof — see footnote 21 of the paper. The first regards a special case of proposition two for a general assumption on the degree of nominal rigidity. For a specific configuration of policy, analytic results are available. The second regards the central bank's imperfect knowledge about the state of the economy. When the central bank can observe current inflation, the Leeper conditions are necessary and sufficient for stability under learning dynamics.

[3] The expressions below are generated by the same Matlab file used for the poof of Proposition 7.

4.1 Arbitrary Nominal Rigidity

Proposition 3 *Assume agents face uncertainty about the monetary and fiscal regimes. If* $\phi_\pi = 0$ *and* $\tau = 0$*, then the rational expectations equilibrium is stable for all parameter values.*

Proof: Assuming $\phi_\pi = 0$ implies that $\lambda_2 = \alpha$. In fact,

$$\begin{aligned}
\lambda_2 &= \frac{1}{2\beta}\left[1+\beta+\kappa(\alpha) - \sqrt{(1+\beta+\kappa(\alpha))^2 - 4\beta}\right] \\
&= \frac{1}{\beta}\left(\frac{1}{2}\beta + \frac{1}{2\alpha}\left(\alpha^2\beta - \alpha\beta - \alpha + 1\right) - \frac{1}{2\alpha}\left(1-\alpha^2\beta\right) + \frac{1}{2}\right) \\
&= \alpha
\end{aligned}$$

The proof proceeds in the same steps as for Proposition 2. The expressions below are obtained from the file[4] *fiscal_ delay_ benchmark.m.* The trace of the $\tilde{A}$ matrix describing the dynamics of the intercept becomes

$$tr(\tilde{A}) + 1 = -\frac{\left[\beta^2\alpha^2 - \alpha\left(\beta^2 + 2\beta\right) + 2\right]}{1-\alpha\beta} < 0$$

while the determinant of the matrix $\tilde{A}$ is equal to -1 for every parameter value. Consider the B matrix (dynamics of the b coefficients): the traces is

$$tr(\tilde{B}) + 1 = \frac{\alpha^4\beta^3 - \alpha^4\beta^2 - \alpha^3\beta^3 - \alpha^3\beta^2 + 2\alpha^2\beta + 2\alpha\beta - 2}{(1-\alpha\beta)\left(1-\alpha^2\beta\right)} = \frac{G(\alpha)}{(1-\alpha\beta)\left(1-\alpha^2\beta\right)}.$$

Consider the numerator $G(\alpha)$. It is straightforward to show that

$$G(0) = -2 \text{ and } G(1) = -2\beta^3 - 2\beta^2 + 2\beta + 2\beta - 2 < 0.$$

Finally,

$$\begin{aligned}
G'(\alpha) &= -4\alpha^3\beta^2(1-\beta) - 4\alpha^2\beta^2 + \alpha^2\beta^2 + 4\alpha\beta + 2\beta - 3\alpha^2\beta^3 = \\
&\quad \left[-4\alpha^3\beta^2(1-\beta) + 4\alpha\beta\left(1-\alpha\beta\right)\right] + \alpha^2\beta^2 + 2\beta - 3\alpha^2\beta^3 \\
&> \left[-4\alpha^3\beta^2(1-\beta) + 4\alpha\beta\left(1-\alpha\beta\right)\right] + \alpha^2\beta^2 + 2\alpha^2\beta^2 - 3\alpha^2\beta^3 \\
&= \left[-4\alpha^3\beta^2(1-\beta) + 4\alpha\beta\left(1-\alpha\beta\right)\right] + 3\alpha^2\beta^2 - 3\alpha^2\beta^3 > 0
\end{aligned}$$

showing that the trace is always negative. The determinant is always equal to -1.

[4] In order to generate the result, select the option "peg=1".

4.2 Resolving Central Bank Uncertainty

Proposition 4 *Assume the central bank can perfectly observe current inflation. Then the stability conditions under learning coincide with the conditions for local determinacy.*

The proof follows the same steps as Proposition 2. The expressions below are obtained by using the file *fiscal_current_analytical.m.* Under perfect information on the part of the central bank, *both* matrices A and B can be further reduced by exploiting an extra restriction on their parameters. For matrix A,

$$A_{3,j} = \phi_\pi A_{2,j} \;\text{ for } j = 1...5$$

and the same for matrix B. The reduced matrices $\tilde{A}$ and $\tilde{B}$ are in this case only two dimensional. Real negative eigenvalues require positive determinant and negative trace.

Ricardian regime. As $\alpha \to 0$, trace and determinant of matrix $\tilde{A}$ converge to

$$\lim_{\alpha\to 0} tr\left(\tilde{A}\right) = -\frac{2-\beta}{1-\beta} + \frac{1}{(1-\beta)\,\phi_\pi} < 0$$

and

$$\lim_{\alpha\to 0} \det\left(\tilde{A}\right) = \frac{\phi_\pi - 1}{\phi_\pi\,(1-\beta)} > 0$$

provided $\phi_\pi > 0$. For the matrix $\tilde{B}$, the trace is

$$tr(\tilde{B}) = \frac{-\phi_\tau\,(1-\beta)\,(\beta\phi_\pi + 1) - \beta\phi_\pi + 1}{\phi_\pi\beta(1-\beta)\phi_\tau}$$

which can be verified t be negative if $\phi_\tau > 1$ (as required in the Ricardian regime) and $\phi_\pi > 1$. Finally, the determinant is

$$\det(\tilde{B}) = \frac{-\phi_\tau\,(1-\beta) + 1 - \beta\phi_\pi}{\phi_\pi\beta(1-\beta)\phi_\tau}$$

is negative if $\phi_\tau > 1$ (as required in the Ricardian regime) and $\phi_\pi > 1$.

Non-Ricardian **regime**.Solving for the determinant of $\tilde{A}$ we get

$$\lim_{\alpha\to 0} \det(\tilde{A}) = \frac{1-\phi_\tau}{1-(1-\beta)\,\phi_\tau} > 0$$

Values of ϕ_τ consistent with a non-Ricardian fiscal rule satisfy

$$-1 < H(\phi_\tau) = \frac{\beta}{1-(1-\beta)\phi_\tau} < 1.$$

Multiplying the determinant by β (which leaves its sign unchanged) we get that the condition to be satisfied is

$$\frac{\beta(1-\phi_\tau)}{(1-\phi_\tau+\phi_\tau\beta)} > 0. \tag{12}$$

for $\phi_\tau < 1$ and $\phi_\tau > \frac{1+\beta}{1-\beta}$. Consider the trace. We obtain

$$\begin{aligned}\lim_{\alpha\to 0} tr(\tilde{A}) &= -\frac{(\phi_\tau\beta + 2 - 2\phi_\tau)}{(1-\phi_\tau+\phi_\tau\beta)} \\ &= -\left[1 + \frac{1-\phi_\tau}{(1-\phi_\tau+\phi_\tau\beta)}\right] < 0.\end{aligned}$$

Concerning the matrix $\tilde{B}$, the trace is

$$\lim_{\alpha\to 0} tr(\tilde{B}) = G^T(\phi_\pi) - \frac{(\beta^2\phi_\pi\phi_\tau - \beta\phi_\pi\phi_\tau + 2\beta\phi_\pi - 2\phi_\tau\beta + 2\phi_\tau - 2)}{(\beta\phi_\pi - 1)(1-\phi_\tau+\phi_\tau\beta)}.$$

In the case $\phi_\pi = 0$ we get

$$G^T(0) = -\frac{2(1-\phi_\tau+\phi_\tau\beta)}{(1-\beta)(1-\phi_\tau+\phi_\tau\beta)} = -\frac{2}{(1-\beta)} < 0$$

and for $\phi_\pi = 1$,

$$G^T(1) = -\frac{(\beta^2\phi_\tau - \beta\phi_\tau + 2\beta - 2\phi_\tau\beta + 2\phi_\tau - 2)}{(\beta-1)(1-\phi_\tau+\phi_\tau\beta)} = -\left[1 + \frac{1-\phi_\tau}{(1-\phi_\tau+\phi_\tau\beta)}\right] < 0$$

Last,

$$\begin{aligned}G'^T(\phi_\pi) &= \frac{\beta\phi_\pi(\beta-1)}{(1-\phi_\tau+\phi_\tau\beta)^2(\beta\phi_\pi-1)} \\ &= \frac{\beta\phi_\pi(1-\beta)}{(1-\phi_\tau+\phi_\tau\beta)^2(1-\phi_\pi\beta)} \geq 0 \text{ for every } \phi_\pi \in [0,1]\end{aligned}$$

which implies that the trace is negative for every value of ϕ_π and ϕ_τ consistent with the determinate and stationary REE. The determinant,

$$\lim_{\alpha\to 0} \det(\tilde{B}) = \frac{1-\phi_\pi\beta - (1-\beta)\,\phi_\tau}{(1-\phi_\pi\beta)(1-\phi_\tau\,(1-\beta))} > 0$$

for $\phi_\tau < 1$ and $\phi_\tau > \frac{1+\beta}{1-\beta}$.

5 Learning to Believe in the Fiscal Theory: An Example

To illustrate the stability properties of the non-Ricardian equilibrium under learning, consider a deterministic economy with fully flexible prices; fiscal policy characterized by zero steady state debt, $\delta = 0$, and an exogenous constant surplus, $\phi_\tau = 0$; and a central bank with perfect information about inflation so that $i_t = \phi_\pi\pi_t$. Under these assumptions, aggregate supply equals the natural rate of output, and the model is given by the aggregate demand and debt equations

$$\phi_\pi \hat{\pi}_t = (1-\beta\phi_\pi)\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\hat{\pi}_{T+1} \tag{13}$$

$$\hat{b}_{t+1} = \beta^{-1}\left(\hat{b}_t - \hat{\pi}_t\right). \tag{14}$$

Let beliefs be specified by the regressions $\hat{\pi}_t = \omega_\pi \hat{b}_t + \varepsilon_{\pi,t}$ and $\hat{b}_{t+1} = \omega_b \hat{b}_t + \varepsilon_{b,t}$. For simplicity assume that the intercept is not estimated. The belief structure implies

$$\begin{aligned} \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\hat{\pi}_{T+1} &= \omega_\pi \frac{\omega_b}{1-\beta\omega_b}\hat{b}_{t+1} \\ &= \omega_\pi \frac{\omega_b}{1-\beta\omega_b}\left[\beta^{-1}\hat{b}_t - (\beta^{-1}-\phi_\pi)\hat{\pi}_t\right] \end{aligned} \tag{15}$$

where the second equality uses the definition of the flow budget constraint. Inserting (15) in (13) and rearranging provides

$$\begin{aligned} \hat{\pi}_t &= \left[\phi_\pi \frac{1-\beta\omega_b}{\omega_\pi\omega_b} + (1-\beta\phi_\pi)(\beta^{-1}-\phi_\pi)\right]^{-1}(1-\beta\phi_\pi)\beta^{-1}\hat{b}_t \\ &= T(\omega_\pi, \omega_b)\,\hat{b}_t \end{aligned}$$

which denotes the actual evolution of inflation as a function of real debt and agents' beliefs.

In the special case $\phi_\pi = 0$, where monetary policy is a nominal interest rate peg, the expression simplifies to

$$\hat{\pi}_t = \hat{b}_t \tag{16}$$

and observed dynamics are independent of agents' beliefs. Indeed, relation (16) corresponds to the restriction between inflation and debt that obtains in a rational expectations equilibrium under the maintained assumptions. Given $T(\omega_\pi, \omega_b) = 1$, the associated ordinary differential equations characterizing learning dynamics are

$$\dot{\omega}_\pi = 1 - \omega_\pi \text{ and } \dot{\omega}_b = -\omega_b,$$

implying stability for all parameter values.

More generally, stability under learning depends on the relation between inflation and government debt. Suppose agents' inflation expectations increase for unmodelled reasons — formally $\omega_\pi > 1$. The increase in inflation expectations leads to an increase in current inflation, with the increase being larger the smaller is ϕ_π. Simultaneously, higher inflation decreases the real value of next-period holdings of the public debt, which in turn lowers expectations. In the limiting case, $\phi_\pi \to 0$, inflation remains unchanged — the two effects on inflation are equal and opposite. In the more general case, with $0 < \phi_\pi < 1$, the initial rise in inflation expectations is not validated by subsequent inflation data and the agents' estimate of ω_π converges back to its rational expectations equilibrium value. As long as agents' beliefs permit a possible relation between inflation and real debt, as assumed in this paper, their learning process converges to rational expectations equilibrium.

6 Alternative Models of Learning Dynamics

Many recent papers have proposed analyses of learning dynamics in the context of models where agents solve infinite-horizon decision problems, but without requiring that agents make forecasts more than one period into the future. In these papers, agents' decisions depend only on forecasts of future variables that appear in Euler equations used to characterize rational expectations equilibrium. Important contributions include Bullard and Mitra (2002) and Evans and Honkapohja (2003). Of most relevance to the present study is Evans and Honkapohja (2007) which similarly studies the interaction of monetary and fiscal policy, but in a model of learning dynamics in which only one-period-ahead expectations matter to expenditure and pricing plans of households and firms. The following section replicates part of their analysis in the context of the model developed here, and contrasts the resulting findings with those of sections 5 and 6.

Since the optimal decision rules for households and firms presented in section 2 are valid under arbitrary assumptions on expectations formation, they are satisfied under the rational expectations assumption. Application of this assumption implies the law of iterated expectations to hold for the aggregate expectations operator and permits simplification of relations (10) and (11) in the paper to the following aggregate Euler equation and Phillips curve:[5]

$$\begin{aligned}\hat{x}_t &= E_t\hat{x}_{t+1} - (\hat{\imath}_t - E_t\hat{\pi}_{t+1} - r_t) \\ \hat{\pi}_t &= \kappa\hat{x}_t + \beta E_t\hat{\pi}_{t+1}.\end{aligned}$$

Under learning dynamics, with only one-period-ahead expectations, it is assumed that aggregate demand and supply conditions are determined by

$$\hat{x}_t = \hat{E}_t\hat{x}_{t+1} - \left(\hat{\imath}_t - \hat{E}_t\hat{\pi}_{t+1} - r_t\right) \quad (17)$$

$$\hat{\pi}_t = \kappa\hat{x}_t + \beta\hat{E}_t\hat{\pi}_{t+1}. \quad (18)$$

[5] See Preston (2005a, 2005b) for a detailed discussion.

Identical assumptions are made on monetary and fiscal policy provide the remaining model equations

$$\hat{\imath}_t = \phi_\pi E^{cb}_{t-1}\hat{\pi}_t \quad (19)$$

$$\hat{s}_t = \phi_\tau \hat{b}_t \quad (20)$$

and

$$\hat{b}_{t+1} = \beta^{-1}\left(\hat{b}_t - \hat{\pi}_t - (1-\beta)\,\hat{s}_t\right) + \hat{\imath}_t \quad (21)$$

The model is closed with a description of beliefs. As nominal interest rates and taxes need not be forecast, an agent's vector autoregression model is estimated on the restricted state vector

$$X_t = \begin{bmatrix} \hat{x}_t \\ \hat{\pi}_t \\ \hat{b}_{t+1} \end{bmatrix}.$$

Two points should be underscored. First, the assumption that only one-period-ahead forecasts matter, implies that households and firms do not take account of transversality conditions in making their spending and pricing plans. Decisions are not optimal given maintained beliefs. That households fail to make decisions that satisfy their intertemporal budget constraint might be thought to have implications in the present context as the fiscal theory of the price level is a theory grounded on shifting evaluations of various variables related precisely by this constraint. Furthermore, by ignoring the implications of the intertemporal budget constraint, fiscal policy has no direct impact on spending and pricing decisions. Neither forecasts of future taxes nor the average indebtedness of the macroeconomy matter for aggregate dynamics. Second, and related, is that because households do not need to forecast future nominal interest rates or taxes there is no uncertainty about the policy rules adopted by the monetary and fiscal authority — there is no regime uncertainty and no role for communication of the joint policy strategy. It seems worth exploring the consequences of these alternative modeling assumptions, and learning whether they elucidate earlier results.

In the model given by relations (17), (18), (19), (20) and (21) the following stability results obtain.

Proposition 5 *For $0 < \alpha < 1$, stabilization policy ensures expectational stability if and only if the following conditions are satisfied: either*

1. *Monetary policy is active and fiscal policy is locally Ricardian such that*

$$\phi_\pi > 1 \text{ and } 1 < \phi_\tau < \frac{1+\beta}{1-\beta} \text{ ; or}$$

2. Monetary policy is passive and fiscal policy is non-Ricardian such that

$$0 \leq \phi_\pi < 1 \text{ and either } 0 \leq \phi_\tau < 1 \text{ or } \phi_\tau > \frac{1+\beta}{1-\beta}.$$

This generalizes the Evans and Honkapohja (2006) analysis to a model with nominal rigidities.[6] When only one-period-ahead expectations matter, the Leeper conditions are sufficient to rule out expectations-driven instability. In contrast, in a model of optimal decisions, these conditions obtain only if there is no regime uncertainty — i.e. the policy rules are credibly communicated to households and firms — and either agents believe the government accounts to be intertemporally solvent or the fiscal authority chooses policy so that the steady state debt-to-output ratio is zero. If neither of these conditions is met, the analysis of this paper suggests a smaller menu of policies is consistent with expectations stabilization. Furthermore, economies with non-zero debt-to-output ratios experience rather different dynamics in response to disturbances — recall the impulse response functions of the previous section.

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[6] The proof is available on request.