Abstract

I study how the informational effect of monetary policy leads to gains from commitment. Monetary policy has an informational effect when the private sector has imperfect information about the underlying economy and extracts information about unobserved shocks from the central bank’s interest-rate decisions. With serially uncorrelated shocks, I show that the optimal monetary policy rule responds more aggressively to natural-rate shocks and less aggressively to cost-push shocks, relative to the central bank’s optimizing response under discretion. The optimal policy rule improves ex-ante welfare by reducing the information revealed on cost-push shocks, which consequently reduces the stabilization bias caused by actual cost-push shocks under perfect information. In addition, I study how external information and serial correlation in shocks affect the size of gains from commitment. A calibrated dynamic model shows that, with relatively precise external information, committing to the optimal rule improves ex-ante welfare by 54 percent compared with the equilibrium outcome under optimizing discretionary policy.
1 Introduction

It has become widely accepted that the effect that monetary policy has on the economy depends on the beliefs held by the private sector. Past literature has shown the gains from commitment when beliefs in the private sector can be optimally controlled by the central bank when it commits to policy rules.\(^1\) While the importance of expectation is well established, previous literature has only studied the case in which commitment changes the expectations on the policy itself. In this paper, I study how gains from commitment can come from the informational effect of monetary policy. Monetary policy has an informational effect when the central bank has better information than the private sector about the state of the economy. Consequently, the private sector can extract information about the underlying shocks from the changes in the interest rate. I demonstrate that the central bank can change how beliefs are formed in the private sector about different shocks by committing to a state-contingent policy rule, which leads to welfare gains from commitment.

The informational effect of monetary policy builds on the assumption of information friction in the private sector. Previous literature has studied both the case where the central bank is better informed about relevant economic fundamentals than the private sector, and the case where the central bank has less precise information than the private sector does. With a few exceptions, the majority of these papers assume that the expectations formed in the private sector about the underlying state of the economy are independent of monetary policy decisions. However, recent empirical papers have demonstrated that changes in the interest rate also affect the beliefs in the private sector about economic fundamentals.\(^2\) In this paper, I study how the optimal conduct of monetary policy should be affected by the central bank’s awareness of the information that will be revealed by its policy decisions. In particular, I examine whether policy rule achieves welfare gains as opposed to discretionary policy.

I build a New Keynesian model with Calvo price rigidity and information frictions in the private sector. There are two types of shocks: natural-rate shocks and cost-push shocks. Due to imperfect information, the equilibrium output gap and inflation depend on both the actual shocks and the beliefs about the shocks, along with the interest rate decisions by the central bank.

The central bank is assumed to have perfect information about both types of shocks. It sets the interest rate conditional on the actual shocks to minimize its loss function given by the weighted sum of squared inflation and output gap. Private agents with rational expectation correctly understand the reaction function of the interest rate, so that they regard the interest rate as a public signal.

\(^1\) See Kydland and Prescott (1977), Barro and Gordon (1983), Clarida, Gali and Gertler (2000), Woodford (1999), Eggertsson et al. (2003), among others. A more comprehensive review on gains from commitment is provided in the literature review session.

\(^2\) See Campbell et al. (2012) and Nakamura and Steinsson (2013) as examples of empirical studies on the informational effect of monetary policy.
which provides information simultaneously about the two shocks. In this situation, the interest rate has two effects on the equilibrium in the private sector: the traditionally studied direct effect on the cost of borrowing for consumers and the informational effect on the beliefs in the private sector.

A central bank can either be discretionary or commits to an interest-rate rule. The discretionary central bank optimizes interest-rate decisions at any given state of the economy and takes the informational effect of its interest-rate decisions as exogenous. In comparison, the central bank with commitment can change the beliefs in the private sector by announcing and committing to a state-contingent rule. When choosing the ex-ante policy rule, the commitment central bank internalizes the informational effect of its interest-rate decisions and balance between the direct effect and the informational effect of the interest rate.

To study how the optimal rule differs from the equilibrium interest-rate decision of the discretionary central bank, I start with the simple case where shocks have no serial correlations. Private agents are rational. They correctly understand how interest rates react to both shocks, but have imperfect information about the shocks. Private agents form beliefs through a Bayesian updating process, where they regard the interest rate set by the central bank as a signal to extract information about the two shocks. When the interest rate reacts positively to both shocks, it becomes one signal that provides information jointly to the two shocks. When the private sector forms expectation about one shock, the prior distribution of the other shock becomes the source of noise as the interest rate being the signal. I demonstrate that beliefs formed through a Bayesian updating process are more sensitive to the shock to which the interest rate responds more aggressively or has a higher ex-ante dispersion.

The informational effect of the interest rate applies differently to the equilibrium output gap and inflation. Under the assumption that shocks have no serial correlation and the assumption that interest rate only responds to shocks in current period, the expectations about future equilibrium variables are at their steady state levels. Consequently, even though private agents are forward-looking, expectations about future equilibrium do not matter in their decisions in current period. I assume that the consumer is able to observe current price levels, but each individual firm do not observe the aggregate price level. Consequently, the output gap is free from the effect of beliefs. Yet the inflation depends on the beliefs in the private sector as optimal pricing decisions are strategic complements, where the resetting price of each firm also depends on the firm’s expectation about the aggregate price level. Thus, the interest rate changes the output gap only through the direct effect, but affects inflation through both the direct effect and the informational effect. When the central bank reacts to expansionary shocks by increasing the interest rate, the informational

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3I use the term "expansionary shocks" to refer to the shocks that cause positive output gap or inflation without the response of interest rates. That is, positive natural-rate shocks (negative current TFP shocks) and positive cost-push shocks.
effect dampens the direct effect of the increase in the interest rate, as the private sector updates its beliefs about the expansionary shocks.

To compare the optimizing discretionary policy with the optimal policy rule, I first look at how the Phillips curve is changed by the informational effect of the interest rate. The Phillips curve is the constraint that a central bank faces, which captures the co-movement of the output gap and the inflation as a result of changes in interest rates. After a marginal increase of the interest rate, the direct effect on the household’s cost of borrowing decreases both the output gap and the inflation, which results in a positively sloped Phillips curve under perfect information. However, with imperfect information, as the informational effect dampens the direct effect on the inflation, the Phillips curve becomes flatter than that under perfect information. In addition, under perfect information, a cost-push shock induces a positive intercept to the Phillips curve, as a cost-push shock increases the inflation only without changing the natural output level. This positive intercept of the Phillips curve results in the stabilization bias, which is the conflict between the closing the output gap and minimizing the inflation. Under perfect information, a central bank increases the interest rate to partially offset the effect of the cost-push shock on inflation, which results in a positive inflation and a negative output gap. However, with imperfect information, as the private sector simultaneously updates beliefs about both the cost-push shock and the natural-rate shock from a positive change in the interest rate, the intercept caused by the cost-push shock is reduced. For the same reason, although a natural-rate shock does not result in a positive intercept under perfect information, it does so under imperfect information.

In the case of a discretionary central bank, I solve for the Markov perfect equilibrium between the central bank and the private sector. The private sector forms beliefs and makes optimal consumption and pricing decisions while expecting the central bank to play the equilibrium optimizing interest rate at any state of the economy. The central bank optimizes the interest rate to minimize the deviations of inflation and output gap from their target, taking as given the informational effect of its interest-rate decision. A discretionary central bank does not internalize the change of informational effect when making interest-rate decisions.

The change in the Phillips curve under imperfect information leads to a change in the optimizing discretionary monetary policy in equilibrium. Although the natural-rate shock can be completely offset by discretionary monetary policy under perfect information, this "divine coincidence" cannot be achieved with information frictions. This is because even if the actual shock is a natural-rate shock, the private sectors still assigns a positive possibility to the event that the interest rate is reacting to a cost-push shock. Consequently, optimizing discretionary policy is "leaning against the wind" after both shocks, seeking a negative correlation between output gap and inflation. I show that the optimizing discretionary interest rate reacts more to natural-rate shocks and less to cost-push shocks than what is optimal under perfect information.
Second, I show the gains from committing to the optimal policy rule. To isolate the informational gains, I focus on the case where the interest rate only responds to current shocks. This removes the traditionally studied gains of commitment to a delayed response, where the gains from commitment come from the change in expected future equilibrium. I show that even without the traditionally studied effect on expected future equilibrium, the optimal policy rule still improves ex-ante welfare compared with the equilibrium under discretion, as the policy rule optimally controls the information revealed about the unobserved shocks.

In the case of commitment, the central bank controls the informational effect by announcing and committing to a state-contingent interest rate rule. I demonstrate that relative to the optimizing discretionary interest rate, the optimal interest rate rule responds more aggressively to natural-rate shocks and less aggressively to cost-push shocks. When the private sector believes that interest rate are less sensitive to the cost-push shocks, beliefs about the cost-push shocks are less sensitive to changes in the interest rate. Consequently, both the slope and the intercept of the Phillips curve are endogenously determined by the optimal decision of policy rule. The optimal policy rule improves ex-ante welfare, because it reduces the marginal increase in the expected cost-push shock after a marginal increase in the interest rate, which consequently reduces the stabilization bias caused by the actual cost push shock under perfect information.

The informational effect of monetary policy results in a novel time inconsistency problem. Different from the traditional time inconsistency, in which the incentives to deviate apply across time, the time inconsistency problem in my model applies across states. Once the central bank has committed to a policy rule, it has fixed the informational effect of the interest rate, and thus has fixed the Phillips curve. Ex-post, the central bank has an incentive to deviate from its committed rule, assuming such change of interest-rate response will not change the Phillips curve. Suppose that there is a positive natural-rate shock, then prior to the realization of the shock, the central bank commits to react more aggressively relative to the optimizing response under discretion, in order to reduce the informational effect on the expected cost push shock. This policy rule reduces the intercept of the Phillips curve after the natural rate shock. After the Phillips curve is fixed, the central bank wants to reduce the increase in the interest rate, assuming such one-time deviation from the rule will not change the Phillips curve after the natural rate shock.

I extend the analysis in two ways. First, I incorporate external signals, which models any other source of information obtained by the private agents, including direct communication by the central bank. The external signals are distributed independently around the actual shocks. Without the informational effect of the interest rate, increasing the precision of signal to one shock only makes the expected shock closer to the actual shock ex-ante. However, with the presence of the informational effect of the interest rate, the effect of external signals is not independent. When increasing the precision of the external signal to one shock, it also makes the interest rate a more precise sig-
nal to the other shock. Consequently, this interaction effect yields different welfare implications for central bank communication from the conventional wisdom. Providing more precise information about the efficient shock (natural-rate shock) through central bank communication may reduce welfare if at the same time the private sector also knows more precisely about the inefficient shock (cost-push shock) from the interest rate.

Second, I extend the analysis to serially correlated shocks to study the dynamic informational effect of the interest rate. In this case, the dynamic informational effect of current interest rate comes from the persistent beliefs formation process in the private sector. The private agents form beliefs in the current period by optimally combining current signals and past beliefs. Consequently, the current interest rate has a lagged effect on future equilibrium through its effect on current beliefs. When the central bank considers the dynamic effect of its interest rate decisions, the objective function of a discretionary central bank includes deviations of output gap and inflation in both current and future periods. The optimal discretionary policy can be characterized as "dynamically leaning against the wind": it is willing to tolerate a positive sum of current inflation and current output gap if the sum of inflation and output gap in the future is expected to be negative.

To quantify the gains from commitment, I calibrate the full version of my model, including external signals, serially correlated shocks, and policy implementation errors. In my calibrated model, I adopt parameter values from previous macroeconomics literature, except for the precision of external information. Varying the precision of external information critically changes the size of gains from commitment. In the extreme case where external information is infinitely imprecise, the gains from commitment are negligible. However, when external signals are as precise as actual shocks, optimal policy rule can improve welfare by 54 percent relative to the equilibrium under optimizing discretionary policy.

1.1 Related Literature

My paper connects three strands of literature: (i) comparison of monetary policy under discretion versus commitment, (ii) optimal monetary policy under information frictions, and (iii) informational effect of monetary policy.

(A) Discretionary Monetary Policy versus Monetary Policy Rule

There has been a long history of studying the gains from monetary policy commitment. The original treatment can be found in Kydland and Prescott (1977) and Barro and Gordon (1983), who discuss the classical inflationary bias that is a result of discretionary central bank having an objective function that contains a positive target of output gap. A large literature has developed various methods to overcome the inflationary bias under discretion, including central bank reputation (Barro (1986) and Cukierman and Meltzer (1986) etc.) and different central bank preferences.
Another mechanism that leads to gains from commitment is when discretionary central bank faces stabilization bias. This happens when there is a trade-off between closing the output gap and minimizing inflation in current period. By committing to a delayed response of the interest rate, central bank is able to decrease current inflation without sacrificing current output gap, but through decreases in the expected future inflation. Clarida, Gali and Gertler (2000) study how an ad-hoc cost-push shock introduces a conflict between inflation stabilization and output gap stabilization, and describe the optimal commitment to a future path interest rate. Woodford (1999) studies how an interest rate smoothing objective helps central bank to commit to a history-dependent policy, in order to steer private sector expectations on the future policy rate. Eggertsson et al. (2003) show that optimal commitment to delayed response can mitigate the distortions created by the zero lower bound on the interest rate.

(B) Optimal Monetary Policy with Information Frictions

My paper builds on the studies of optimal monetary policy under imperfect information. This field is revived since Woodford (2001), which shows how higher order beliefs lead to persistent effect of monetary policy, under the assumption of imperfect information which is initially introduced in Phelps (1970) and Lucas (1972).

The majority of papers which study the optimal monetary policy under information frictions assume that beliefs in the private sector are formed independently from monetary policy decisions, and that central bank makes policy decisions every period taking as given the exogenous beliefs in the private sector. Ball, Mankiw and Reis (2005) assume information is rigid in the private sector, and characterize optimal policy as an elastic price standard. Adam (2007) assumes an endogenous learning process in the private sector, and demonstrates the target of optimal monetary policy changes from output gap stabilization to price stabilization when information becomes more precise. Angeletos and La’O (2011) solve the Ramsey problem for optimal monetary policy and show that the flexible-price equilibrium is no longer first-best when information frictions affect real variables.

There are also papers that discuss the gains from policy commitment under imperfect information. Svensson and Woodford (2003) and Svensson and Woodford (2004) assume that the central bank has imperfect information, and show that the optimal policy under commitment displays considerable inertia, relative to the discretionary policy, due to the persistence in the learning process. Lorenzoni (2010) and Paciello and Wiederholt (2013) explore the idea that the central bank is able to change the learning process in the private sector if it is able to commit to completely offset the inefficient shocks.

Recent papers have started to investigate the situation in which the private sector extracts information about the underlying economy from monetary policy decisions. Baeriswyl and Cornand
(2010) point out that as markup shocks cannot be fully neutralized by monetary policy, central bank alters optimal policy response in order to reduce the information revealed on cost push shock through policy decisions. Berkelmans (2011) demonstrates that with multiple shocks, tightening policy may initially increase inflation. The paper closest to mine is Tang (2013), which shows that when the private sector has rational expectation, the stabilization bias is reduced when monetary policy has information effect.

The the best of my knowledge, the only paper that discusses the time inconsistency problem due to the informational effect of monetary policy is Stein and Sunderam (2016). The authors use a reduced-form model in which the central bank balances between implementing the optimal target rate and minimizing information revelation about this target. In their paper, private agents are assumed not to have rational expectation on the central bank’s behaviors. The discretionary central bank always has incentives to deviate from the target interest rate, in order to reveal less information about its target. In my paper, I assume that private agents have rational expectation on how central bank would react for both discretionary policy and policy rule. Relative to the perfect information case, both optimizing discretionary policy and optimal policy rule exhibit inertial response to cost-push shocks, and the degree of inertia is higher under commitment in order to optimally control the informational effect of the interest rate.

(C) Empirical Evidence on the Informational Effect of Monetary Policy

My study is also motivated by the empirical evidence on the informational effect of monetary policy. Romer and Romer (2000), Romer and Romer (2004) are the first ones to provide empirical evidences on information asymmetry between the Federal Reserve and the private sector. They show that inflation forecasts by private agents respond to changes in policy-rate after FOMC announcements. Faust, Swanson and Wright (2004) further confirm that the private sector revises its forecasts in response to monetary policy surprises. In more recent papers, Campbell et al. (2012) show that unemployment forecasts decrease and CPI inflation forecasts increase after a positive innovation to future federal funds rates. Nakamura and Steinsson (2013) identify the informational effect of federal funds rate through high frequency data. In addition, Melosi (2016) captures this empirical pattern using a DSGE model with dispersed information. Garcia-Schmidt (2015) uses Brazilian Survey data of daily forecasts data and show that inflation forecasts in the private sector increase in the short run after an unexpected tightening policy.

The remainder of the paper is organized as follows. Section 2 characterizes the optimization decisions by the representative household in the private sector, and expresses aggregate output gap and inflation as functions of beliefs. Section 3 analyzes optimizing discretionary policy and gains from commitment to policy rule in the baseline case where shocks are not serially correlated. Section 4 and section 5 discuss two factors that affect the size of gains from commitment: external information and serial correlation in shocks. To quantitatively assess the gains from commitment,
I calibrate the full version of my model with serially correlated shocks, external signals and policy implementation error in section 6. Section 7 concludes the paper.

2 Private Sector

In this section, I incorporate information frictions to an otherwise standard New Keynesian model with Calvo-type price rigidity. Fluctuations are driven by two types of shock which are: a technology shock (expressed in terms of "Wicksellian natural rate" in output gap) and a wage markup shock (expressed in terms of cost push shock in inflation). I assume that the central bank has perfect information on the two shocks whereas the private sector cannot observe the shocks directly. The private sector has rational expectations on the central bank’s behavior. In particular, the private sector correctly understands how central bank will respond to both shocks, and infers information about the shocks from observing the interest rate decision. This section describes the equilibrium level of aggregate output gap and inflation as functions of beliefs in the private sector.

2.1 Information Frictions

Following Phelps (1970), Woodford (2001), and Angeletos and La’O (2010), I model an "island economy", where the information friction is the result of geographical isolation. There is a continuum of islands, indexed by \( j \), and a representative household. The household consists of a consumer and a continuum of workers. In the beginning of each period, the household sends one worker to each island \( j \). There is a continuum of monopolistic firms, each located on one island and indexed by \( j \) as the index of the island. Each firm demands labor in the local labor market on its island, and produces a differentiated intermediate good. Information is symmetric within island, as each firm is able to observe its firm-specific shocks. Information is asymmetric across islands, as firms are unable to observe shocks or decisions by other firms. Consequently, the resetting price of each firm depends on the firm’s expectation about the aggregate price level, which makes aggregate inflation as a function of beliefs in the private sector. The consumer of the representative household makes inter-temporal consumption decisions. He is able to observe current prices of all intermediate goods, but unable to observe shocks directly. Consequently, the inter-temporal consumption decisions are also subject to information frictions.
2.2 Private Sector Optimization Problem

2.2.1 Household

The preferences of the representative household are defined over the aggregate consumption good, $C_t$ and the labor supplied to each firm, $N_t(j)$ as

$$E_H^t \sum_{t=0}^{\infty} \beta^t \left\{ U(C_t) - \int V(N_t(j))dj \right\},$$

(1)

where $E_H^t$ denotes the household’s subjective expectations conditional on its information set, $\omega_H^t$. The demand of aggregate good $C_t$ is made up of a continuum of intermediate goods

$$C_t = \left( \int_0^1 C_t(j)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}},$$

(2)

where $C_t(j)$ is the consumption of intermediate good $j$ in period $t$.

The economy is cashless. The household maximizes expected utility subject to the inter-temporal budget constraint:

$$\int P_t(j)C_t(j)dj + B_{t+1} \leq \int W_t(j)N_t(j)dj + (1+i_t)B_t + \Pi_t,$$

(3)

where $B_t$ is a risk-free bond with nominal interest $i_t$, which is determined by the central bank. $\Pi_t$ is the lump-sum component of household income which includes tax payments and profits from all firms. $W_t(j)$ and $N_t(j)$ is the labor wage and labor supply in firm $j$, correspondingly.

The household’s optimization problem can be solved in two stages. First, conditional on the level of aggregate consumption, the household allocates intermediate goods consumption to minimize the cost of expenditure conditional on the level of aggregate good consumption. The allocation of intermediate good consumption that minimizes expenditure yields:

$$C_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{\frac{1}{1-\epsilon}} C_t,$$

(4)

where $P_t = \left[ \int_0^1 P_t(j)^{1-\epsilon}dj \right]^{\frac{1}{1-\epsilon}}$.

In the second stage, given the aggregate price level, $P_t$, the household decides aggregate consumption, $C_t$, labor supply to all firms, $N_t(j)\forall j$, and saving in risk-free bond, $B_{t+1}$. I assume the utility of aggregate good consumption and the utility of labor supply take the form: $U(C_t) = C_t^{\frac{1}{1-\sigma}}$, and $V(N_{jt}) = N_{jt}^{\frac{1}{1-\phi}}$, where $\sigma$ is the inverse of inter-temporal elasticity of substitution and the parameter $\phi$ is the inverse of the Frisch elasticity of labor supply. The inter-temporal consumption
decision leads to the Euler equation:

$$C_t^{−\sigma} = \beta (1 + it) E_{t}^{H} \left( C_{t+1}^{−\sigma} \frac{P_t}{P_{t+1}} \right).$$  \hspace{1cm} (5)$$

Equation (5) shows that consumption decisions are forward-looking. Current demand depends on the relative cost of consumption today versus consumption tomorrow.

The intra-temporal labor supply decision sets the marginal rate of substitution between leisure and consumption equal to the real wage:

$$\frac{N_t^{\phi}(j)}{C_t^{−\sigma}} = W_t$$ \hspace{1cm} (6)

2.2.2 Firms

Firms make two decisions to maximize expected profits: the intra-period cost minimization and the optimal pricing decision. As the cost minimization problem only involves information within the island and information is symmetric within islands, the intra-period cost minimization problem is free from any information frictions. The optimal pricing decision, on the other hand, is affected by both Calvo price rigidity and the information friction. In each period, a measure $1 - \theta$ of firms gets the Calvo lottery to reset their prices. Other firms charge their previous prices. A firm $j$ that resets its price in period $t$ chooses $P_t^*(j)$ to maximize its own expectation of the sum of all discounted profits while $P_t^*(j)$ remains effective. The profit optimization problem can be written as:

$$\max_{P_t^*(j)} \sum_{k=0}^{\infty} \theta^k E_t^j \left\{ Q_{t,t+k} \left[ P_t^*(j) Y_{t+k}(j) - U_t^w(j) W_{t+k}(j) N_t(j) \right] \right\},$$ \hspace{1cm} (7)

where $E_t^j$ denotes the firm $j$’s expectation conditional on its information set, $\omega_t$. $Q_{t,t+k}$ is the stochastic discount factor defined by: $Q_{t,t+k} = \beta^k \frac{U'(C_{t+k})}{U'(C_t)} \frac{P_t}{P_{t+k}}$. $U_t^w(j)$ denotes the wage markup for firm $j$.

Firms face two constraints. The first one is the demand of their products, which results from the household’s optimal allocation among intermediate goods. The second one is the production technology. For the purpose of tractability, and in alignment with the tradition in the New Keynesian literature, I assume that labor is the only input and each firm produces according to a constant return to scale technology,

$$Y_t(j) = A_t(j) L_t(j),$$ \hspace{1cm} (8)

where $A_t(j)$ denotes the technology of firm $j$.

There are two sources of uncertainty that affect the pricing decisions of each firm: technology shocks and wage markup shocks. I assume that both shocks have an aggregate component and an
idiosyncratic component. The idiosyncratic components are drawn independently in every period, and are distributed log-normally around their aggregate components.

\[
\log(A_t(j)) \equiv a_t(j) = a_t + s_t^a(j), \quad s_t^a(j) \sim N(0, \sigma_{sa}^2)
\]

\[
\log(U_t^w(j)) \equiv u_t^w(j) = u_t^w + s_t^u(j), \quad s_t^u(j) \sim N(0, \sigma_{su}^2)
\]

I assume that the aggregate components of both shocks follow AR(1) processes:

\[
a_t = \phi^a a_{t-1} + \nu_t^a, \quad \nu_t^a \sim N(0, \sigma_{va}^2)
\]

\[
u_t^w = \phi^u u_{t-1}^w + \nu_t^{uw}, \quad \nu_t^{uw} \sim N(0, \sigma_{uw}^2)
\]

The first order condition for labor input implies that the nominal marginal cost of production is \(U_t(j)W_t(j)/A_t(j)\). Substituting the marginal cost of production into the optimal pricing decision of firm \(j\) results in

\[
P_t^*(j) = \frac{\epsilon}{\epsilon - 1} \frac{E_t^j \Sigma (\beta \theta)^k u'(C_{t+k})P_{t+k}^e Y_{t+k} \frac{u_{t+k}(j)w_{t+k}(j)}{A_{t+k}(j)}}{E_t^j \Sigma (\beta \theta)^k u'(C_{t+k})P_{t+k}^e Y_{t+k}}.
\]

Equation (9) implies that individual resetting prices are forward looking and strategic complements. It increases with the expectation of a higher marginal cost of production and a higher aggregate price level in both current and all future periods.

### 2.3 Aggregation and Equilibrium in the Private Sector

Equilibrium variables in the private sector are solved in log deviations from steady state values (i.e., \(x_t \equiv \ln(X_t/X)\)), and denoted by lower-case letters. (See Appendix A for details).

**The Output Gap**

Following the New Keynesian tradition, I express output in terms of the output gap \(\hat{y}_t\), which is defined as the difference between \(y_t\) and the natural level of output \(y_t^n\). The natural level of output is defined to be the output level under flexible prices and perfect information. In such situation, \(y_t^n\) becomes a linear function of \(a_t\): \(y_t^n = \frac{\phi + \sigma}{1 + \phi} a_t\), and follows an AR(1) process, \(y_t^n = \phi y_{t-1}^n + \nu_t\), where \(\phi = \phi^a\), and \(\sigma = \frac{\phi + \sigma}{1 + \phi} \sigma_{va}\).

The output gap is derived as follows:

\[
\hat{y}_t \equiv y_t - y_t^n = E_t^{H} \hat{y}_{t+1} - \frac{1}{\sigma} \left[ i_t - \left( \frac{1}{1 - \phi} r_t^n - \frac{\phi}{1 - \phi} E_t^{H} r_t^n \right) - E_t^{H} r_{t+1} \right],
\]

where \(E_t^{H} \hat{y}_{t+1} = E_t^{H} y_{t+1} - E_t^{H} y_t^n = E_t^{H} y_{t+1} - \phi E_t^{H} y_t^n\). \(r_t^n\) denotes the natural rate of interest,
which is the equilibrium real interest rate that equates output to its natural level under perfect information and flexible prices. It is calculated as \( r^n_t \equiv \sigma (E_t y_{t+1} - y^n_t) = \sigma (\phi - 1) y^n_t \).

If information is perfect, \( E^H_t r^n_t = r^n_t \), and expectations about future equilibrium are objective i.e., \( E^H_t \hat{y}_{t+1} = E_t \hat{y}_{t+1} \) and \( E^H_t \pi_{t+1} = E_t \pi_{t+1} \). Substitute them into the above equation results in the IS curve under perfect information:

\[
\hat{y}_t = E_t \hat{y}_t - \frac{1}{\sigma} [i_t - r^n_t - E_t \pi_{t+1}] \tag{11}
\]

The difference between equation (10) with equation (11) illustrates how output gap under imperfect information differs from that under perfect information. Specifically, under perfect information, a positive natural rate shock increases the output gap by \( \frac{1}{\sigma} r^n_t \). The positive output gap is due to the price rigidity, as there is insufficient increase in the aggregate price level, which results in less reduction in the output level than the reduction in natural output level. In comparison, this output gap is enlarged under imperfect information. Without the response of the interest rate, the private agents do not update beliefs about the natural rate. Substituting \( E^H_t \pi_{t+1} = 0 \) into equation (10) shows that the output gap becomes \( \frac{1}{1-\phi} \frac{1}{\sigma} r^n_t \). Intuitively, as the household does not know about the change of natural output level in the next period, the household does not reduce current consumption at all, which is represented by a larger positive output gap.

**Inflation**

According to the assumption of Calvo-type price rigidity, the current aggregate price level is the composite of the aggregate price in the previous period and the average resetting prices:

\[
p_t = \theta p_{t-1} + (1-\theta) \int p^*_t(j) \, dj. \tag{12}
\]

The integral of resetting prices potentially leads to the higher order beliefs problem. As equation (9) shows, \( p^*_t(j) \) includes firm \( j \)'s expectation on the aggregate price level \( P_t \), and thus includes other firms’ expectations. This leads to the infinite regress problem, in which each firm uses its firm-specific shock as a private signal, and guesses the private signals observed by other firms. As the focus of my study is on aggregate variables instead of on the distribution of prices across firms, I shall abstract from this higher order beliefs problem by modeling homogeneous subjective beliefs.\(^4\) This means that when all private agents, including both firms and the household, form expectations about the aggregate variables, all agents use only public signals. Therefore, the information sets are the same across all agents. I denote the homogeneous subjective beliefs in the

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\(^4\)There are many papers that address how higher order beliefs leads to monetary policy to have more persistent effect, for example Woodford (2001), Angeletos and La'O (2009). For solution method to the infinite regress problem, see Huo and Takayama (2015), Melosi (2016) and Nimark (2017), as examples.
private sector as $E_t^s$. Mathematically, I assume that the idiosyncratic components of firm-specific shocks have infinite variance. In this case, private signals are completely uninformative, so firms do not use their private signals of firm-specific shocks to form beliefs on aggregate variables.

The aggregation of individual resetting prices leads to the New Keynesian Phillips curve under subjective beliefs (see Appendix A for detailed derivation):

$$\pi_t = \beta \theta E_t^s \pi_{t+1} + (1 - \theta)E_t^s E_t^{\pi} + \kappa \theta \hat{\gamma} + u_t,$$

where $\kappa = \frac{(1-\beta \theta)(1-\theta)(\varphi + \sigma)}{\theta}$, and $u_t$ denotes the cost push shock, which is related to the wage markup shock as $u_t = (1-\theta)(1-\beta \theta)u_t^w$.

If information is perfect, the expected inflation is the same as the actual inflation, i.e., $E_t^s \pi_t = \pi_t$, and expectations about future equilibrium is objective i.e., $E_t^s \pi_{t+1} = E_{t+1} \pi_{t+1}$. Substitute them into equation (13) results in the Phillips curve under perfect information:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{\gamma} + \frac{1}{\theta} u_t$$

The difference between equation (13) and equation (14) shows how the inflation under imperfect information differs from that under perfect information. Under perfect information, a positive cost-push shock increases the inflation by $\frac{1}{\theta} u_t$. As this cost-push shock does not increase the output gap, the central bank faces a conflict between the inflation stabilization and the output gap stabilization. If it increases the interest rate to dampen inflation, then doing so also creates a negative output gap. When information is imperfect, only a fraction $\theta$ of the actual cost push shock is observed by individual firms as their firm-specific shocks. Without the response of the interest rate, each firm does not update beliefs on the aggregate cost push shock, so that resetting prices change by less than under perfect information. Therefore, the imperfect information reduces this stabilization bias under perfect information.

### 3 Monetary Policy with Serially Uncorrelated Shocks

I start by comparing discretionary monetary policy versus policy rule commitment in a simple scenario, in which underlying shocks have no serial correlation. In this case, although private agents are forward-looking, the expectations of future equilibrium variables do not matter for cur-

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5Note that subjective expectation in this paper refers to the rational expectation formed only as a result of imperfect information about the state variables.

6Another way to generate homogeneous beliefs is to assume that firms have the same technology and face the same wage markup, but do not observe them when setting prices. This assumption, however, makes aggregate inflation consists of only the firms’ expectation, and does not consist of actual shocks. Consequently, there will not be inflation versus output gap trade-off due to the lack of actual cost push shock, and optimal monetary policy becomes trivial.
rent choices, because future equilibrium variables are expected to be at their steady state levels. In addition, I shut down the gains from committing to a delayed response, as I impose the restriction that the central bank can only respond to current states. These two assumptions allow me to focus on the within-period gains from commitment through the informational effect of monetary policy.

### 3.1 Equilibrium under an Arbitrary Interest Rate Policy

This section studies how interest rates affect output gap and inflation through both the direct effect on the borrowing cost and the informational effect on beliefs. In addition, it illustrates how the informational effect on beliefs about different shocks are determined by the interest rate reaction function.

First, apply the assumption that shocks have no correlation to the IS function and the Phillips curve by substituting \( \phi = 0 \) and \( E_t^s \hat{y}_{t+1} = E_t^s \pi_{t+1} = 0 \):

\[
\hat{y}_t = -\frac{1}{\sigma} (i_t - r_t^n) \quad (15)
\]

\[
\pi_t = (1 - \theta) E_t^s \pi_t + \kappa \theta \hat{y}_t + u_t \quad (16)
\]

As shown in the IS equation, the output gap is free from subjective beliefs and thus the informational effect of interest rate does not apply in determining the output gap. This is because future equilibrium variables are expected to be at steady state levels and current aggregate price level are observed by the consumer.

On the other hand, inflation is affected by subjective beliefs, as individual firms do not observe aggregate price level when deciding optimal resetting prices. Consequently, to express the actual inflation in terms of shocks, further substitute the expected aggregate inflation by \( E_t^s \pi_t = \kappa E_t^s \hat{y}_t + \frac{1}{\theta} E_t^s u_t \). The expected output gap is different from the actual output gap, as the private sector has imperfect knowledge on the actual \( r_t^n \). Specifically, \( E_t^s \hat{y}_t = \hat{y}_t - \frac{1}{\sigma} r_t^n + \frac{1}{\sigma} E_t^s r_t^n \). As a result, inflation can be expressed in terms of the output gap, the actual and the expected shocks as:

\[
\pi_t = \kappa \hat{y}_t + (1 - \theta) \frac{\kappa}{\sigma} (E_t^s r_t^n - r_t^n) + \frac{1 - \theta}{\theta} E_t^s u_t + u_t \quad (17)
\]

The interest rate has two effects on equilibrium in the private sector. The first one is the direct effect, which is the conventionally studied effect on the borrowing cost for the household. The direct effect of a marginal increase in the interest rate reduces current consumption, because it increases the relative cost of current consumption versus future consumption. In addition, the direct effect of an increase in the interest rate also reduces the aggregate price level, as each firm

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7In align with conventional New Keynesian literature, the long run distortion has been taken care of via Pigouvian tax as an employment subsidy, so that the steady state levels of output gap and inflation are all zero.
reduces its resetting price when facing with a lower demand. The direct effect of the interest rate on the output gap and the inflation are:

\[ \frac{\partial \hat{y}_t}{\partial i_t} |_{direct} = -\frac{1}{\sigma}, \quad \frac{\partial \pi_t}{\partial i_t} |_{direct} = \frac{\partial \pi_t}{\partial \hat{y}_t} \frac{\partial \hat{y}_t}{\partial i_t} = -\frac{\kappa}{\sigma}. \]

The informational effect captures how the interest rate changes the beliefs in the private sector about the two underlying shocks \( E_{s}^{r} r_{t}^{n} \) and \( E_{s}^{u} u_{t} \). As the output gap is not affected by the subjective beliefs, it is free from the informational effect of the interest rate. The marginal informational effect of the interest rate on the inflation is the combination of the marginal change in the expected cost-push shock and in the expected natural-rate shock. The marginal informational effect of the interest rate on output gap and inflation is:

\[ \frac{\partial \hat{y}_t}{\partial i_t} |_{informational} = 0, \quad \frac{\partial \pi_t}{\partial i_t} |_{informational} = \frac{\partial \pi_t}{\partial E_{s}^{r} r_{t}^{n}} \frac{\partial E_{s}^{r} r_{t}^{n}}{\partial i_t} + \frac{\partial \pi_t}{\partial E_{s}^{u} u_{t}} \frac{\partial E_{s}^{u} u_{t}}{\partial i_t}. \]

where the partial derivatives of inflation on the expected natural-rate and the expected cost-push shock are defined in equation (17) as: \( \frac{\partial E_{s}^{r} r_{t}^{n}}{\partial i_t} = (1 - \theta) \frac{\kappa}{\sigma}, \frac{\partial E_{s}^{u} u_{t}}{\partial i_t} = \frac{1 - \theta}{\theta} \).

**State and Signals**

To study the informational effect of the interest rate, one first needs to specify the (unobserved) state variables and the signals about the state variables. As shown in the IS curve and the Phillips curve, only the aggregate part of the shocks matter in the determination of the output gap and the inflation. In addition, technology shocks and wage markup shocks can be written in terms of natural-rate shocks and cost-push shocks, \( r_{t}^{n} \) and \( u_{t} \) correspondingly.

\[ r_{t}^{n} = \phi r_{t-1}^{n} + v_{t}, \quad u_{t} = \phi u_{t-1} + v_{t}^{u}, \]

where the natural-rate shock and the cost-push shock are mapped from the technology shock and the wage markup shock as \( r_{t}^{n} = \frac{\phi + \sigma}{1 + \phi} \sigma(\phi - 1) a_{t}, \) and \( u_{t} = (1 - \theta)(1 - \beta \theta)u_{t}^{w}. \)

Denote the auto-coefficients of the natural-rate shock and the cost-push shock as \( \phi \) and \( \phi_{u}. \) By construction, they are the same as the auto-coefficients for the aggregate technology process and the wage markup process. In this section, I assume that both shocks are serially uncorrelated \( (\phi = \phi_{u} = 0) \). Denote the standard deviation of the natural-rate shock and the cost-push shock as \( \sigma_{r} \) and \( \sigma_{u}. \) By construction, \( \sigma_{r} = \phi_{u} \sigma(\phi - 1) \sigma_{\text{v}_a}, \) and \( \sigma_{u} = (1 - \theta)(1 - \beta \theta)\sigma_{u_{w}}. \)

I assume that the private agents have rational expectations on the response function of the interest rate. Under an arbitrary linear interest rate function which responds linearly to the two aggregate shocks, i.e., \( i_{t} = F_{r} r_{t}^{n} + F_{u} u_{t}, \) the interest rate becomes \textit{one signal} that simultaneously
provides information on two shocks.

If there is only one shock to which interest rate responds linearly, the private sector will be able to perfectly infer the actual shock. In this case, the economy becomes no different from the perfect information case. In the case with two shocks, when private agents regard the interest rate as a signal to one shock, the prior distribution of the other shock becomes the source of noise in this signal.

**Beliefs Formation**

Agents in private sector are Bayesian, and form best linear forecasts by optimally weighting their prior beliefs (shocks have zero ex-ante mean) and the current signal (the interest rate). Let $K_r$ and $K_u$ denote the optimal weights on the two states after observing interest rate changes, which are determined through the optimal filtering process. Beliefs formed about the two states obtained through the Kalman Filtering process are

$$
\begin{bmatrix}
E_t^s r_t
\end{bmatrix} = \begin{bmatrix} 1 - K_r & 0 \\ 1 - K_u & 0 \\ \end{bmatrix} \begin{bmatrix} r_t \\ u_t \\ \end{bmatrix} + \begin{bmatrix} K_r \\ K_u \\ \end{bmatrix} \begin{bmatrix} K_r F_r & K_r F_u \\ K_u F_r & K_u F_u \\ \end{bmatrix} \begin{bmatrix} r_t \\ u_t \\ \end{bmatrix},
$$

(18)

Eqs. (18) shows that in the solution of the Kalman filtering process with an arbitrary interest rate reaction function, the sensitivity of beliefs to the actual shock is the product of the sensitivity of beliefs to the interest rate ($K_r$ or $K_u$) and the sensitivity of the interest rate to the actual shocks, ($F_r$ or $F_u$). The following lemma provides an interpretation for equation (18).

**Lemma 1:** *Beliefs are more sensitive to the shock which (1) the interest rate responds more aggressively to, and (2) has higher ex-ante dispersion.*

Lemma 1 describes, for a given ex-ante dispersion of the shocks, how the precision of interest rate as a signal is determined by the response function of the two shocks. As private agents in the private sector cannot distinguish whether a change in interest rate responds to the natural rate shock or to the cost push shock, they think interest rate is more likely to respond to the shock to which it is more sensitive. For example, if the interest rate barely responds to cost-push shocks, then after observing a change in the interest rate, agents in the private sector infer that the change 8Another way to maintain imperfect information while having only one state variable is to include an implementation error in the interest rate, so that interest rate becomes a noisy signal. In Section 6 where I quantitative assess the gains from commitment, I also incorporate implementation error.
of interest rate is less likely to be a response to a cost-push shock. Otherwise, given that $F_u$ is very small, the change in the interest rate has to come from a large cost-push shock, which is less likely to realize given the prior distribution of the cost-push shock. On the other hand, for any given interest rate reaction function, agents in the private sector update more towards the shock that has higher ex-ante dispersion, as the ex-ante mean of the shock is less weighted in beliefs formation process.

Notice the difference between the sensitivity of beliefs to actual shocks and the sensitivity of beliefs to interest rate. Lemma 1 suggests that for a given $F_r$, the sensitivity of the expected cost-push shock to the actual cost-push shock $\frac{\partial E_t^{u_t}}{\partial u_t} (K_u F_u)$ increases as $F_u$ increases, but not necessarily the sensitivity of expected cost push shock to interest rate, $\frac{\partial E_t^{i_t}}{\partial i_t} (K_u)$. This is illustrated in the following figure.

![Figure 1: The Informational Effect of Interest Rate (Lemma 1)](image)

In the first row, $F_r$ is fixed at 1.5, and $\sigma_r = \sigma_u = 0.1$. When $F_u$ increases from 0.1 to 3, $K_u F_u$ (right figure) increases monotonically. However, as shown in the left figure, $K_u$ increases first, but then decreases at larger value of $F_u$. In the second row, I hold $F_r = F_u = 2$, and $\sigma_r = 0.1$. Increasing $\sigma_u$ monotonically increases both the sensitivity of beliefs to interest rate and the sensitivity of beliefs (left figure) to the actual shock (right figure).

First, I hold $\sigma_r = \sigma_u = 0.1$, and show the change of sensitivity of the expected cost-push shock
to interest rate ($K_u$) and the sensitivity of the expected cost-push shock to the actual shock ($K_r F_r$) while holding $F_r$ fixed at 1.5. When $F_u$ starts to increase from a small value, both the sensitivity of beliefs to interest rate and the sensitivity of beliefs to the actual shock increases. However, as $F_u$ becomes bigger, the informational effect is dominated by interest rate becoming more sensitive to shocks rather than beliefs being more sensitive to interest-rate changes. As shown in the left figure in the first row, the sensitivity of $E_t^s u_t$ to the change in $i_t$ decreases at higher level of $F_u$. Next, in the second row, I fix $F_r = F_u = 2$, and $\sigma_r = 0.1$, and analyze changes in $\sigma_u$ from 0.01 to 1. Both the sensitivity of beliefs to interest rate and to the actual shock increases.

### 3.2 Discretionary Monetary Policy

In the previous section, I analyzed the informational effect for a given interest rate rule. Here, I analyze the equilibrium between the private sector and the central bank in which the central bank optimizes in a discretionary way. Specifically, central bank plays its best response by setting the interest rate to maximize its objective at any given state, taking as given the informational effect of the interest rate. The private sector has rational expectations, in the sense that it perfectly understands the best response function of the interest rate, and extracts information on the current states through the optimal filtering process. Simultaneously as beliefs are formed, the household chooses consumption and firms set prices optimally.

The optimizing interest rate is an endogenous decision by the central bank, whose objective function consists of equilibrium variables in the private sector. The equilibrium variables in the private sector depend on the beliefs in the private sector, which in turn depends on the equilibrium interest rate reaction function. This introduces circularity into the belief formation problem. The solution of this problem is discussed by Svensson and Woodford (2003). Following their method, I study the optimizing interest rate in equilibrium by first conjecturing an interest rate reaction function, with which private agents form beliefs. Next, I show how the constraint of the discretionary central bank is affected by the informational effect of the interest rate in equilibrium, and then solve for the optimizing interest rate decision under such constraint. Finally, I solve for a Markov perfect equilibrium such that the response of the interest rate is consistent with the previously conjectured interest rate reaction function.

#### 3.2.1 The Phillips Curve

I start the analysis of the discretionary monetary policy by discussing the constraint faced by the central bank, which is the Phillips curve. The Phillips curve captures the trade-off between output gap stabilization and inflation stabilization, as the interest rate changes both the output gap and the inflation. With perfect information, the slope of the Phillips curve is exogenous to the decision of
the interest rate. Also, with perfect information, the Phillips curve crosses the origin of the \((\hat{y}_t, \pi_t)\) plane after a natural-rate shock and has a positive intercept after a cost push shock.

However, with imperfect information, the Phillips curve not only depends on the realization of actual shocks, but also depends on the expectations about the shocks. As the expectations about the two shocks are determined by the reaction function of the interest rate, I first guess a linear reaction function of the interest rate as \(i_t = F_r r^n_t + F_u u_t\). Then, I substitute the expected shocks under such interest rate reaction function, to solve for the Phillips curve with the informational effect of the interest rate:

\[
\pi_t = \kappa \hat{y}_t + \left[ (1 - \theta) \frac{\kappa}{\sigma} K_r + \frac{1 - \theta}{\theta} K_u \right] i_t - (1 - \theta) \frac{\kappa}{\sigma} r^n_t + u_t.
\]

(19)

To express the trade-off between output gap stabilization versus inflation stabilization, I substitute interest rate by its relation with output gap from the IS equation, \(i_t = -\sigma \hat{y}_t + r^n_t\). This results in the Phillips curve with the informational effect of the interest rate

\[
\pi_t = \left\{ \kappa - \sigma \left[ (1 - \theta) \frac{\kappa}{\sigma} K_r + \frac{1 - \theta}{\theta} K_u \right] \right\} \hat{y}_t + \left\{ (1 - \theta) \frac{\kappa}{\sigma} (K_r - 1) + \frac{1 - \theta}{\theta} K_u \right\} r^n_t + u,
\]

(20)

where \(K_r\) and \(K_u\) are determined through the optimal filtering process in equation (18).

For a given interest rate function which responds positively to the two shocks, I plot the Phillips curve under imperfect information, in comparison with the Phillips curve under perfect information in Figure 2.\(^9\)

The following lemma summarizes the differences between the Phillips curve under perfect information and the Phillips curve under imperfect information:

**Lemma 2:** For a given interest rate function that reacts to both shocks in a linear way, the informational effect of the interest rate changes the Phillips curve in three aspects, relative to the Phillips curve under perfect information:

1. The slope of the Phillips curve is flatter than that under perfect information.
2. The intercept after a cost push shock is reduced.
3. There is non-zero intercept after a natural rate shock.

\(^9\)see Section 6 or parameter values
In the above figures, I plot the Phillips curve under discretionary policy while fixing the interest rate reaction function to be $F_r = 1$ and $F_u = 1$. The prior distribution of the two shocks are set equal, $\sigma_r = \sigma_u = 0.1$.

**Proof: see Appendix**

The intuition is the following:

For (1), the slope captures the co-movement between the output gap and the inflation as results
of changes in the interest rate. The informational effect of the interest rate on inflation reduces the co-movement between output gap and inflation. After a tightening monetary policy, the direct effect of the interest rate reduces output gap, as the higher nominal interest rate increases the real cost of borrowing. Under perfect information, the direct effect of on inflation is given by $\kappa$, but under imperfect information, this direct effect is dampened by the informational effect. When observing a higher interest rate, private agents assigns a positive possibility to the event that the interest rate is reacting to a positive cost-push shock. This update in the expected cost-push shock leads to an increase in the expected inflation. This update of beliefs reduces the direct tightening effect of the interest rate on inflation, which reduces the co-movement between output gap and inflation.

For (2), the intercept caused by an actual cost push shock is reduced because information on the actual cost push shocks is only partially revealed through the interest rate. Note that although both the actual cost-push shock and the expected cost-push shock induce an increase in inflation, the expected cost push shock does not cause an intercept. This is due to the fact that when the output gap stays at zero, it indicates the interest rate does not change, which makes expected cost-push shock to be zero. In fact, the effect of expected cost push shock is captured in the slope, rather than the intercept of the Phillips curve.

For (3), after a positive natural rate shock, the intercept of the Phillips curve represents the equilibrium output gap and inflation when when the interest rate tracks the natural rate one by one, $i_t = r_n^t$. Due to the informational effect of the interest rate, the change of the interest rate makes the private agents simultaneously updates beliefs on both shocks, inflation is not zero, and the sign depends on the expected cost-push shock and the difference between the expected natural-rate shock and the actual natural-rate shock. First, the expected cost-push shocks increase inflation, as each firm believe the aggregate price level increases, as each firm believes that other firms have a higher wage markup. Second, as the private agents underestimate the realization of the natural-rate shock, each firm expects aggregate demand to be less than the actual level. Consequently, the negative difference between the expected and the actual natural-rate shock decreases inflation. The relative size of the two effects determines the sign of the intercept.

### 3.2.2 Optimal Discretionary Monetary Policy

When shocks are serially uncorrelated and the interest rate does not respond to lagged variables, the current interest rate does not affect future output gap and inflation. Thus, when choosing the current interest rate, although central bank is forward-looking, the objective function of the central bank consists only deviations of the current output gap and the inflation. The optimization problem
by the discretionary central bank is given by:

$$
\min_{i_t} L(t) = \left[ \pi_t \quad \hat{y}_t \right] \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \left[ \pi_t \quad \hat{y}_t \right] + \text{indept. terms}
$$

(21)

subject to

$$\hat{y}_t = -\frac{1}{\sigma} (i_t - r^*_{it})$$

(22)

$$\pi_t = \kappa \hat{y}_t + (1 - \theta) \frac{\kappa}{\sigma} (E^x_i r^n_t - r^n_{it}) + \frac{1 - \theta}{\theta} u_t$$

(23)

$$E^x_i r^n_t = K_r i_t$$

(24)

$$E^x_i u_t = K_u i_t$$

(25)

where $\omega$ is a constant that results from the second order approximation of the household’s utility.\(^{10}\)

**Definition:** A perfect Markov equilibrium between a discretionary central bank and the private sector with rational expectations can be described in aggregate terms in the following way:

(i) Inflation and output gap result from household’s optimal consumption choices and firms’ optimal price-setting behavior, which are shown in equations (10) and (13);

(ii) Beliefs in the private sector about the realization of shocks are formed through the Kalman Filtering process as shown in equation (18);

(iii) Interest rate is set by the central bank’s constrained optimization problem as specified in (21).

To solve for the equilibrium interest rate, we first need to conjecture an interest rate reaction function, $i_t = F^0 r^n_i + F^0 u_t$, which determines the Phillips curve. Then, the central bank chooses the interest rate to maximize its objective function under the constraint of the Phillips curve. The equilibrium interest rate under rational expectation is found as the fixed point between the conjectured interest rate function and the optimizing interest rate solution. I analyze the characteristics of the optimizing discretionary interest rate in the rest of this section.\(^{11}\)

The first order condition with respect to $i_t$ from equation (21) is given by:

$$\pi_t = - \left( \frac{\partial \pi_t}{\partial i_t} \right)^{-1} \frac{\partial \hat{y}_t}{\partial i_t} \omega \hat{y}_t \equiv -R \hat{y}_t.$$ 

(26)

\(^{10}\)see Woodford (2011) for general derivation of second order approximation of household’s utility under perfect information, and Adam (2007) for application to imperfect information. Appendix D.2 show the derivation that applies to the specific assumption in this paper.

\(^{11}\)Detailed derivation of solving equilibrium optimizing interest rate is provided in Appendix B.
Lemma 3: When shocks are serially uncorrelated, discretionary monetary policy seeks a negative correlation between current output gap and inflation after both natural-rate shocks and cost-push shocks. The absolute value of the correlation coefficient is greater than that under full information.

The intuition of this result is the following: As the intercept is generally not zero after both shocks (Lemma 2), the interest rate increases after positive realizations of both shocks. As the optimizing central bank chooses the tangent point between its indifference curve \( L(\hat{y}_t, \pi_t) \) and the Phillips curve, the equilibrium \((\hat{y}^*_t, \pi^*_t)\) vector is orthogonal to the Phillips curve. As the Phillips curve has a smaller slope with the information effect, the resulting vector of \((\pi^*_t, \hat{y}^*_t)\) becomes steeper.

More explicitly, under full information, the absolute value of the correlation between output gap and inflation is

\[
R_{\text{perfect info}} = \left( \frac{\partial \pi_t}{\partial i^*_t} \right)^{-1} \frac{\partial \hat{y}_t}{\partial i^*_t} \omega = \left( -\frac{\kappa}{\sigma} \right)^{-1} \left( -1 \right) \omega. \tag{27}
\]

With information frictions, the marginal effect of the interest rate on inflation is dampened by the informational effect, so

\[
R_{\text{imperfect info}} = \left( \frac{\partial \pi_t}{\partial i^*_t} \right)^{-1} \frac{\partial \hat{y}_t}{\partial i^*_t} \omega = \left( -\frac{\kappa}{\sigma} + (1 - \theta) \frac{\kappa}{\sigma} K_r + \frac{1 - \theta}{\theta} K_u \right)^{-1} \left( -1 \right) \omega. \tag{28}
\]

Under usual parameter values, the interest rate responds positively to both shocks, i.e., \(F_r > 0, F_u > 0\). Therefore, the Kalman gains are positive, \(K_r > 0, K_u > 0\), which results in \(R_{\text{imperfect info}} > R_{\text{perfect info}}\).

We now turn into finding the equilibrium interest rate that achieves the target which is specified in Lemma 3.

First recall that the equilibrium interest rate tracks one-to-one with the change in natural rate, as doing so completely closes the output gap and stabilizes the inflation. The optimal response to cost-push shock is "leaning against wind", which results in \(\pi_t = -\frac{\omega}{\kappa} \hat{y}_t\).

Denote the equilibrium interest rate under discretionary central bank and perfect information as \(i_t = F^p_r r^p_t + F^p_u u_t\), where

\[
F^p_r = 1, \quad F^p_u = \left( \frac{\kappa}{\sigma} \right)^{-1} \frac{\sigma}{\theta}.
\]

Denote the equilibrium interest rate of discretionary monetary policy under imperfect information as \(i_t = F^d_r r^d_t + F^d_u u_t\). The following assumptions help me compare the equilibrium discretionary interest rate with imperfect information and with perfect information.
Assumption 1: 
\[(1 - \theta) \frac{\kappa}{\theta} (K_r(F_p^r, F_u^p) - 1) + \frac{1 - \theta}{\theta} K_u(F_p^r, F_u^p) > 1, \]
where $K_r$ and $K_u$ denotes the Kalman gains for updating beliefs on expected natural-rate and cost-push shock as specified in equation (18).

Assumption 2: $R^p > \bar{R}$, where $R^p$ represents the absolute value of the correlation between output gap and inflation, which is given by equation (23).12

Lemma 4: Under Assumption 1 and 2, the equilibrium discretionary interest rate reacts more aggressively to natural-rate shocks and less aggressively to cost-push shocks under imperfect information, relative to its equilibrium response under perfect information, i.e., $F_d^r > F_p^r$ and $F_d^u < \frac{1}{\theta} F_u^p$.

Proof: see Appendix

Assumption 1 guarantees that under imperfect information, when the central bank implements $(F_p^r, F_u^p)$, the Phillips curve has a positive intercept after natural-rate shock. In this situation, as suggested by Lemma 3, the discretionary central bank should increase the interest rate to achieve a negative output gap, which is equivalent to a $F_d^r$ that is greater than $F_p^r$.

There are two factors that drive the change in the optimal response to cost-push shocks under imperfect information, as both the slope and the intercept of the Phillips curve is changed by the informational effect. First, as suggested by Lemma 2, the intercept after a cost-push shock reduces from $\frac{1}{\theta} u_t$ under perfect information to $u_t$ under imperfect information. Holing the slope constant, this reduction proportionally reduces the decrease in the equilibrium output gap and the equilibrium response of the interest rate. Second, holding the intercept fixed, Assumption 2 dictates that the change in the slope also results in an increase in the equilibrium output gap. Therefore, the two factors result in a smaller response of the interest rate to a cost-push shock, $F_d^u < \frac{1}{\theta} F_u^p$.

3.3 Monetary Policy Rule

In contrast to the problem of a discretionary central bank, which takes as given how beliefs will be formed in the private sector, a commitment central bank is able to control beliefs by announcing a monetary policy rule prior to the realization of shocks. After shocks realize, the central bank perfectly observes the shocks and implements the interest rate implied by the rule. Private agents observe the interest rate, form beliefs about the realized shocks and simultaneously choose consumption and pricing decisions.

With forward looking agents, expectations on future equilibrium matter for current consumption and pricing decisions. Consequently, even with serially uncorrelated shocks, a commitment central bank may choose a policy rule that responds to past shocks, so that the expectations about the direct effect of future interest rates also change the current equilibrium, which potentially leads to gains from committing. The gains from committing to a delayed response still apply under im-

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12See appendix for specific expression for $\bar{R}$
perfect information. However, in order to focus on the within-period gains from the informational effect, I study a state-contingent policy rule that only responds to current shocks.

As the expectation on the interest rate response function determines the beliefs formed about the two shocks, by announcing a state-contingent policy rule, a central bank also chooses a direct mapping from actual shocks to the beliefs about those shocks. In this section, I show that the Phillips curve becomes endogenous to the choice of the policy rule. In addition, I show that there is always a profitable deviation from the optimal policy rule, if the private sector keeps believing that central bank will follow the rule.

3.3.1 Phillips Curve

For a commitment central bank, the Phillips curve is no longer an exogenous trade-off between output gap and inflation. Commitment central bank internalizes the fact that its policy-rule decisions will change the marginal informational effect of the interest rate, and consequently changes the trade-off between inflation and the output gap.

Specifically, the Phillips curve applying to a central bank with commitment that describes the available trade-off between output gap and inflation is given by

$$\pi_t = \left\{ \kappa - \sigma \left[ (1 - \theta) \frac{\kappa}{\sigma} K_r^* + \frac{1 - \theta}{\theta} K_u^* \right] \right\} \hat{y}_t + \left\{ (1 - \theta) \frac{\kappa}{\sigma} (K_r^* - 1) + \frac{1 - \theta}{\theta} K_u^* \right\} r^n_t + u,$$

where $K_r^*$ and $K_u^*$ are no longer constant, but endogenously determined by the choice of policy rule, i.e., $K_r^* = K(F_r, F_u), K_u^* = K(F_r, F_u)$ as specified in equation (18).

I plot the Phillips curve under commitment central bank in Figure 3. In the first figure, the blue line illustrates the Phillips curve under perfect information, after a natural-rate shock such that $r^n_t = 1$ and $u_t = 0$. It crosses the origin and has a positive slope of $\kappa$. The red curve represents the Phillips curve. Tracing Phillips curve from a positive output gap to a negative output gap corresponds to an increase in the positive response of the interest rate to the natural-rate shock, which is equivalent to an increasing $F_r$. Importantly, the effect of the increasing $F_r$ to inflation also depends on the value of $F_u$, as $(F_r, F_u)$ jointly determines $(K_r, K_u)$. Therefore, I fix $F_u = 1$ in order to illustrate the effect of the change in $F_r$. When $F_r$ increases, its marginal effect on output gap is constant, $-\frac{1}{\sigma}$, but its marginal effect on inflation changes, because the marginal informational effect of the interest rate changes as $(K_r, K_u)$ changes with respect to $F_r$.

In this figure, this is illustrated by a steeper slope as the output gap decreases. Intuitively, as the private agents expect the interest rate to respond more aggressively to the natural-rate shock, they assign less probability to the event that a cost-push shock realizes. Therefore, the informational effect of the tightening monetary policy leads to less increase in the expected inflation, which results in a steeper slope.
In the above figures, I plot the Phillips curve under policy rule. The first figure shows the Phillips curve after a natural-rate shock, where I fixed $F_u = 1$ and vary $F_r$. The second figure shows the Phillips curve after a cost-push shock, I fixed $F_r = 1$ and vary $F_u$. Prior distribution of shocks are set equal to each other, such that $\sigma_r = \sigma_u = 0.1$.

In the second figure, the blue line is the Phillips curve under perfect information after a cost push shock, such that $r_t^p = 0$ and $u_t = 1$. The red curve is the Phillips curve under imperfect...
information, where I vary the value of $F_u$ while fixing $F_r = 1$. In this figure, the change in the marginal informational effect is more significant than the first figure. Tracing the Phillips curve from left to right, it represents a decreasing response of interest rate to cost push shock, which is equivalent to a decrease in $F_u$. As Lemma 1 suggests, a smaller $F_u$ increases $K_r$, and has a non-monotonic effect on $K_u$. The combined effect depends on the value of $F_u$, together with other parameters. When $F_u$ is very small, an increase in its value decreases output gap, but barely decreases inflation. This is because at this level of $F_u$, an increase in its value increases both $K_r$ and $K_u$, (see Figure 1, top row) so that the informational effect almost completely offsets the direct effect which increases the borrowing cost.

3.3.2 Optimal Policy Rule

The optimal simple rule is found by choosing the interest rate feedback rule $i_t = f(r^n_t, u_t, \pi_t, \hat{y}_t)$ prior to the realization of shocks, which becomes $i_t = F_r r^n_t + F_u u_t$ in equilibrium. The optimal simple rule is found by choosing $F_r$ and $F_u$ to minimize central bank’s ex-ante loss over the state space:

$$
\min_{F_r, F_u} \int \int \pi_t^2(r_t, u_t) + \omega \hat{y}_t^2(r_t, u_t) dr_t^n du_t,
$$

subject to

$$
\hat{y}_t = -\frac{1}{\sigma} \left[ (F_r - 1) r^n_t + F_u u_t \right],
$$

$$
\pi_t = \left\{ -\frac{\kappa}{\sigma} (F_r - 1) + (1 - \theta) \frac{\kappa}{\theta} (K_r F_r - 1) + \frac{1 - \theta}{\theta} K_u F_r \right\} r^n_t
$$

$$
+ \left\{ -\frac{\kappa}{\sigma} F_u + (1 - \theta) \frac{\kappa}{\theta} K_r F_u + \frac{1 - \theta}{\theta} K_u F_u + 1 \right\} u_t
$$

$$
E_t^x r^n_t = K_r F_r r^n_t + K_r F_u u_t \tag{33}
$$

$$
E_t^x u_t = K_u F_u r^n_t + K_u F_u u_t. \tag{34}
$$

Comparing this problem with the problem of a discretionary central bank (equation 21), we find that the available set of combination of $(\hat{y}_t, \pi_t)$ is expanded due to the additional degree of freedom, i.e. instead of choosing interest rate, commitment central bank chooses $F_r$ and $F_u$. By committing to a state-contingent rule that is different from $i_t = F_r^d r^n_t + F_u^d u_t$, the central bank chooses a direct mapping from the actual shocks to the expected shocks. In comparison, even if a discretionary central bank changes its response of interest rate, the private sector still expects that it would follow its equilibrium response, which is described by $(F_r^d, F_u^d)$, so that the informational effect of the interest rate cannot be changed when the central bank does not have credible commitment. In other words, $(K_r, K_u)$ is only endogenously determined by the central bank when it has credible...
commitment. Otherwise, it is exogenous to the central bank and is determined by the private agents’ beliefs on the equilibrium interest rate response function.

To illustrate how the control of the informational effect of the interest rate changes the available trade-off between the output gap and the inflation, consider the case where \( r_t = 1 \) and \( u_t = 0 \). Suppose that the central bank commits to completely unresponsive to the cost-push shock, i.e. \( F_u = 0 \), then the private agents assign probability 1 to the event that the natural-rate realizes when observing a change in the interest rate, such that \( E_t r^n_t = 1 \) and \( E_t u_t = 0 \). In this case, information on both shocks are perfectly revealed through the interest rate, and the economy goes back to the perfect information case where the divine coincidence holds. In comparison, without credible commitment, the private agents do not believe that had a cost-push shock realize, the central bank will not respond by a positive interest rate. Consequently, the private agents still assign positive probability to the event that a cost-push shock realize even if the increase in the interest rate is completely due to the realization of the natural-rate shock. The updates in the expected cost-push shock breaks down the divine coincidence.

However, although such policy rule is desirable after natural rate shock, it is not desirable in all states, as not responding to cost push shock is clearly not optimal had cost-push realize. This is because when an actual cost push shock induces positive inflation, if the interest rate does not increase, it results in positive inflation and zero output gap, contradicting to Lemma 3. The optimal monetary policy balances between the optimal informational effect and the optimal direct effect across all states.

We now turn into comparing the optimal policy rule with the equilibrium interest rate under discretionary central bank by the first order conditions. The key factor in this analysis is that the effect of the interest rate after one shock also depends on how it would react to the other shock, because the informational effect, \((K_r, K_u)\) is jointly determined by the response of the interest rate to both shocks, \((F_r, F_u)\). \((F_r, F_u)\) are jointly determined by the first order condition on \( F_r \) after \( r^n_t \) shock, and the first order condition on \( F_u \) after \( u_t \) shock.

The first order condition on \( F_r \) after \( r^n_t \) shock is:

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u + \Omega_r \frac{\partial K_r}{\partial F_r} F_r + \Omega_u \frac{\partial K_u}{\partial F_r} F_r = \frac{\omega \hat{y}_t}{\sigma \pi_t} \tag{35}
\]

where \( \Omega_r = (1 - \theta) \frac{\kappa}{\sigma} \), and \( \Omega_u = \frac{1 - \theta}{\sigma} K_u \).

Similarly, the first order condition on \( F_u \) after \( u_t \) shock is:

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u + \Omega_r \frac{\partial K_r}{\partial F_r} F_r + \Omega_u \frac{\partial K_u}{\partial F_r} F_r = \frac{\omega \hat{y}_t}{\sigma \pi_t} \tag{36}
\]
In Figure 4, I draw the optimal $F^*_r$ at varying $F_u$ which is the solution to equation (35), and the optimal $F^*_u$ for varying $F_r$ which is the solution to equation (36). The point where two lines cross is $(F^*_r, F^*_u)$.

![Solution for Optimal Commitment Rule](image)

**Figure 4: Solution for Optimal Commitment Rule**

In this figure, each point on the red line represents solution of $F^*_r(F_u)$ that satisfies the first order condition on $F_r$ as specified in equation (35), and each point on the blue line represents solution of $F^*_u(F_r)$ that satisfies the first order condition on $F_u$ as specified in equation (36). The point where two line cross defines $(F^*_r, F^*_u)$.

To illustrate the difference between the equilibrium interest rate under discretion central bank and the optimal policy rule, I write the first order condition on interest rate for discretionary central bank in terms of $F_r$ after $r^n_t$ shock, and $F_u$ after $u_t$ shock.

\[-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u = \frac{\omega y_t}{\sigma \pi_t}\]

\[-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u = \frac{\omega y_t}{\sigma \pi_t}\]

where $\Omega_r = (1 - \theta) \frac{\kappa}{\sigma}$, and $\Omega_u = \frac{1 - \theta}{\sigma} K_u$.

Comparing the first order conditions under commitment, it can be noticed that the discretionary central bank regards the informational effect of the interest rate as exogenous to its interest rate decisions. More precisely, it does not internalize the change in the Kalman gain with respect
to a change in the interest rate. This is because \((K_r, K_u)\) are determined by the private sector’s expectations about the interest rate reaction function. Then, if the private agents believe that the central bank will optimize at any given state, they believe the interest rate will follow \(i_t = F_r^d r_t^n + F_u^d u_t\).

As suggested by Lemma 1, both the sensitivity of the interest rate to shocks, and the prior distribution of shocks matter for the informational effect of interest rate. Next, I post assumptions on the prior distribution of shocks, which help me compare the optimal policy rule with the equilibrium response of the interest rate under discretionary central bank.

**Assumption 3.** \(\sigma_r = \sigma_u\).

**Proposition 1:** Under Assumption 1, 2, and 3, i) optimal policy rule responds more aggressively to a natural-rate shock than the equilibrium response of interest rate under discretionary policy, for a given response to cost-push shocks, and ii) the optimal policy rule responds less aggressively to a cost push shock than the equilibrium response of interest rate under discretionary policy, for a given response to natural-rate shocks.

*Proof:* see Appendix.

Under the above assumptions, increasing the response to natural rate shocks \((F_r)\) or decreasing the response to cost-push shocks \((F_u)\) from the equilibrium response under discretion \((F_r^d, F_u^d)\), decreases the informational effect of the interest rate. Specifically, it means (1) \(\Omega_r \frac{\partial K_r}{\partial F_r} + \Omega_u \frac{\partial K_u}{\partial F_r} < 0\), and (2) \(\Omega_r \frac{\partial K_r}{\partial F_u} + \Omega_u \frac{\partial K_u}{\partial F_u} > 0\).

As the central bank with commitment internalize the effect of the interest-rate decisions on the Phillips curve, it wants to reduce the marginal informational effect of interest rate, making interest rate more "effective" in offsetting the shocks. Assumption (1) guarantees that a higher value of \(F_r\) decreases the marginal informational effect of interest rate after natural-rate shocks, and assumption (2) guarantees that a lower value of \(F_u\) decreases the marginal informational effect of interest rate after cost-push shocks.

The gains from commitment comes from (a) the increase in the slope of the Phillips curve after both shocks and (b) the decrease in the intercept after natural-rate shocks. As shown in the Phillips curve expressed in equation (29), (a) and (b) are equivalent, and thus have same implication on the value of \(F_r^c\) and \(F_u^c\). Intuitively, more precise information on natural rate shocks and less precise information on cost-push shocks brings reduces the conflict between the direct effect and the informational effect of the interest rate.

Another way to investigate the comparison between optimal policy rule and the equilibrium interest rate under discretionary policy is that as central bank with commitment internalize the informational effect of the interest rate, it balances between the optimal informational effect and the optimal direct effect on the borrowing cost.

The optimal informational effect is such that the central bank reveals perfect information about
the natural-rate shock, and completely withhold information about the cost-push shock. This is because the natural-rate shock is efficient, as it changes the natural level of output together with the price level. Therefore, it does not cause conflict between output gap stabilization and inflation stabilization under perfect information. On the other hand, cost push shock only changes price level without chancing the natural level of output. So it is inefficient, and leads to a conflict between output gap stabilization and inflation stabilization.\footnote{Existing literature has discussed how information on efficient shocks is beneficial. See Morris and Shin (2002), Angeletos and Pavan (2007), Angeletos, Iovino and La’O (2016) as examples.} The optimal informational effect of the interest rate can be achieved by either setting $F_r \to \infty$ or setting $F_u = 0$. Balancing the optimal informational effect and the optimal direct effect results in the interest rate being more sensitive to the natural-rate shocks and less sensitive to the cost-push shocks, i.e. $F_r^c > F_r^d$ and $F_u^c < F_u^d$.

### 3.4 Time Inconsistency

In this section, I analyze the time inconsistency problem which refers to the situation where central bank has an incentive to deviate from its previously committed policy rule. The conventional wisdom on the time inconsistency problem applies across time periods. For example, as discussed in Eggertsson et al. (2003), when the current interest rate hits the zero lower bound, the central bank can encourage current consumption by committing to a lower interest rate in future periods such that $E_t \pi_{t+1} > 0$, when consumption decisions are forward-looking. However, the central bank will face a time inconsistency problem at $t+1$, because $\pi_{t+1} > 0$ is sub-optimal. In my baseline model, I have shut down this conventional channel of commitment, so that the time inconsistency across time periods does not apply. Instead, I show a novel time inconsistency problem that applies across states. Specifically, the central bank wants to implement a different interest rate conditional on the realization of shocks, rather than according to its previously announced policy rule. The intuition for the time inconsistency problem is that a discretionary central bank does not take into account the change of the informational effect of interest rates when deviating from the policy rule. Mathematically, if a central bank has convinced the private sector that it will implement $i_t = F_r^c r_t^n + F_u^c u_t$. Then, with the sensitivity of beliefs about the two states to interest rate changes fixed, central bank wants to re-optimize its interest-rate decisions. The incentives of deviation are summarized in the following proposition.

**Proposition 2:** After central bank has committed to a policy rule, there is always a profitable deviation of interest rate after either natural rate shocks or cost-push shocks, as long as such deviation is unexpected and thus the informational effect of the interest rate stays the same.

**Proof:** see Appendix.

The intuition for Proposition 2 is the following. Prior to the realization of shocks, the optimal policy rule has committed to respond more aggressively to natural-rate shocks (Proposition 1), as
it optimally weighs between decreasing the combined informational effect of interest rate and the direct effect on the borrowing cost. If central bank decides to implement an one-time deviation which is not expected by the private sector, it is able to keep the informational effect fixed and considers only its direct effect. By doing this, central bank takes the informational advantage such that the private sector believes the shock is more likely to be a natural-rate shock rather than a cost-push shock, without actually sacrificing a lower output gap when a natural-rate shock realizes. However, if such deviation is anticipated by the private sector, the private sector will update the sensitivity of beliefs to changes in the interest rate, leaving no profitable deviation available for central bank.

I graphically illustrate the incentive of deviation after a natural-rate shock in the following graph.

![The Phillips Curve, $r_t^0 = 1, u_t = 0$](image)

Figure 5: Time Inconsistency Problem

The blue line is the Phillips curve under policy rule, and the red line is the Phillips curve for one-time deviation, assuming $K_r$ and $K_u$ are fixed. The dotted line is the indifference curve of the central bank.

The dotted ellipse curve is the indifference curve for the central bank whose objective function consists the weighed sum of the squared inflation and the squared output gap. As explained in earlier section, the Phillips curve under commitment policy is endogenous to the choice of policy rule. To find $F_t^* (F_{u_t})$, the optimal response to a natural-rate shock under commitment, we first need
to specify the available trade-off between the output gap and the inflation. The blue line represents the available trade-off by varying the value of $F_r$ while fixing $F_u = 1$. $F_r^*$ is chosen as the tangent point between the endogenous Phillips curve and the indifference curve.

At this point, if a central bank deviates from its commitment, but such deviation is not anticipated by the private sector, then the central bank faces a different Phillips curve. This is because the informational effect of the interest rate is fixed by the commitment, so that changing $F_r$ will not change the slope of the Phillips curve. Mathematically, it means that the marginal informational effect of interest rate, $\Omega_r K_r (F_r^c, F_u^c) + \Omega_u K_u (F_r^c, F_u^c)$, will not change as $F_r$ changes, as the private sector expects the central bank to implement $(F_r^c, F_u^c)$.

Specifically, after a natural rate shock, when the central bank deviates to a smaller response of interest rate than it had committed to $(F_r < F_r^c)$, the Phillips curve has a lower $\pi_t$ at any level of $\hat{y}_t$. This change in the Phillips curve suggests that by deviating to a smaller response of interest rate to natural rate shock, central bank achieves a one-time welfare improvement.

4 External Information

In a more realistic setting, the interest rate is not the only signal that the private sector receives about the aggregate state of the economy. In this section, I discuss central bank direct communication, which is an example of the external information in addition to the informational effect of the interest rate. Unlike the informational effect through the interest rate, which is restricted by the signal dimension, central bank direct communication is not bounded by the signal dimension.

Central bank direct communication can be modeled by providing additional signals independently to the actual shocks, and control for the precision of these external signals. The previous literature has discussed about the value of central bank communication, and the general consensus is that more precise information about the efficient shocks (natural-rate shocks in this setting) is welfare improving and more precise information about the inefficient shocks (cost-push shocks in this setting) is detrimental. In my model, however, the value of central bank communication interacts with the informational effect through policy rate.

4.1 Interaction between Information through Policy Rates and through Direct Communication

Denote the external signals through the central bank communication as $m_r^t$ and $m_u^t$, which distribute log normally around the actual shocks, $r_t^n$ and $u_t$. I assume that the interest rate does not react to the external signals. However, the existence of external signals changes the informational effect of

\[14\text{ see Kramer et al. (2008) for survey of literature on central bank communication.}\]
the interest rate, which consequently changes both the equilibrium interest rate under discretionary central bank and the optimal policy rule.

**Signals**

The signals consist both the interest rate and the external signals through the central bank direct communication, which are summarized as:

\[
\begin{bmatrix}
\hat{i}_t \\
\mu_t
\end{bmatrix} = \begin{bmatrix} F_1 & F_3 \\
1 & 0 \\
0 & 1 \end{bmatrix} \begin{bmatrix} \mu_t \\
\epsilon_t \\
\epsilon_u \end{bmatrix} + \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix} \begin{bmatrix} \epsilon_i \\
\epsilon_t \\
\epsilon_u \end{bmatrix}
\]  

(39)

**Beliefs**

The private sector updates beliefs using both the interest rate and the external signals:

\[
\begin{bmatrix}
E_{i_t} \\
E_{\mu_t}
\end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \end{bmatrix} \begin{bmatrix}
\hat{i}_t \\
\mu_t
\end{bmatrix}
\]  

(40)

where the Kalman gains in the \( K \) matrix is determined through the optimal filtering process.

The informational effect of interest rate and the effect of central bank communication interact with each other. When the central bank communicates about one shock more precisely, the private sector assigns a greater weight on the information from direct communication versus the information from the interest rate. At the same time, the interest rate becomes a more precise signal about the other shock. For example, suppose that the central bank precisely communicates about the \( r_t^n \) shock, then after a positive cost-push shock, the private agents know that \( r_t^n = 0 \) through the direct communication by the central bank. In addition, the private agents also observes that the interest rate responds positively, so that they infer precisely that the increase in the interest rate is due to the positive realization of a cost-push shock.

On the other hand, the existence of informational effect through policy rate also changes the effect of central bank communication. \( K_{13} \) and \( K_{22} \) measure how much information is "falsely" updated to beliefs by external signals. Without information effect through policy rate, \( K_{13} \) and \( K_{22} \) equal to zero, since signals are distributed independently, and the signal of one shock does not provide information about the other shock. However, as the interest rate is one signal about the two shocks, the interaction with the informational effect of the interest rate makes central bank unable to separately convey information. Specifically, both \( K_{13} \) and \( K_{22} \) are negative. Intuitively, suppose that interest rate does no change, and external signals on natural rate goes up, people then back out a negative change in the cost-push shock.

\( K_{12} \) and \( K_{23} \) measure how much information is "correctly" updated through external signals. In
Figure 6, I plot how the sensitivity of beliefs to each signal changes when interest rate reacts more aggressively to natural-rate shock, while holding its response to cost-push shock fixed. (Varying \( F_r \) from 0.1 to 2 while fixing \( F_u = 1 \))

![Figure 6: Sensitivity of Beliefs to Different Signals](image)

In all 6 figures, I set \( F_u = 1 \), and vary \( F_r \) from 0.1 to 2. Ex-ante dispersion of shocks are set to be equal with each other, such that \( \sigma_r = \sigma_u = 0.1 \). The top row is the sensitivity of beliefs about natural-rate shocks with respect to all three signals, the interest rate, the external information about the natural-rate shock, and the external information about the cost-push shock. The second row is the sensitivity of beliefs about cost-push shocks with respect to all three signals, the interest rate, the external information about the natural-rate shock, and the external information about the cost-push shock.

As illustrated in the first row, when the sensitivity of \( E_t^r r^n_t \) to the change in the interest rate is not monotonic as suggested by Lemma 1. The second figure in the first row shows the weight on \( m_r^t \) decreases as the interest rate becomes a more precise signal about \( r^n_t \). From the second row, it shows that increasing the sensitivity of interest rate to the natural-rate shock also decreases the sensitivity of \( E_t^r u_t \) to interest-rate changes, and increases the sensitivity of \( E_t^r u_t \) to central bank communication about the cost-push shock. Intuitively, since the private agents optimally weigh these three signals, when interest rate reacts more aggressively to the natural-rate shock, it becomes a more precise signal than \( m_r^t \). At the same time, it becomes a less precise signal than \( m_u^t \). For this
4.2 Value of (External) Information

To assess the value of external information through the direct communication by the central bank, we first need to study how the optimal response of the interest rate changes under discretion and with commitment, as the central bank takes into account the interaction between the informational effect of the interest rate and the direct communication. The Phillips curve with all signals can be found as:

\[
\pi_t = \left\{ \kappa - \sigma \left[ (1 - \theta) \frac{\kappa}{\sigma} K_{11} + \frac{1 - \theta}{\theta} K_{21} \right] \right\} \hat{y}_t + \left\{ (1 - \theta) \frac{\kappa}{\sigma} (K_{11} + K_{12} - 1) + \frac{1 - \theta}{\theta} (K_{21} + K_{22}) \right\} r^n_t + \left\{ (1 - \theta) \frac{\kappa}{\sigma} K_{13} + \frac{1 - \theta}{\theta} K_{23} + 1 \right\} u_t
\]

(41)

The existence of the information effect of the interest rate complicates the welfare effect of central bank communication. Without the information effect of interest rate, welfare is maximized when the central bank provides perfectly precise signal about the efficient shock, the natural-rate shock, and completely uninformative signal about the inefficient shock, the cost-push shock. However, with the information effect of the interest rate, if agents in the private agents has precise information about one shock, they are able to infer precise information about the other shock through the interest rate.

In Figure 1, I illustrate the welfare implication of the direct communication by the central bank. In the first row of Figure 7, I show the contour plot at varying precision of central bank communication under optimizing discretionary policy (left) and under optimal policy rule (right). It shows that when communication about the cost push shock becomes more imprecise, which is modeled by a lower \( \sigma_{eu} \), the ex-ante loss increases. This is in align with the conventional wisdom that more precise information about the inefficient shock is welfare reducing. On the other hand, when increasing the precision of central bank communication about natural rate shock, which is modeled by a lower \( \sigma_{er} \), the ex-ante loss also increases. This contradicts the conventional wisdom.

In summary, when interest rate is able to provide sufficiently precise information about the efficient shock, additional direct communication about either shocks reduces ex-ante welfare.

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For conventional wisdom on value of information, see Morris and Shin (2002), Angeletos and Pavan (2007), Angeletos, Iovino and La’O (2016), for examples
Next, I add an implementation error to the interest rate function, such that \( i_t = F_r r_t^N + F_u u_t + e_t \). The interest rate becomes a noisier signal of both shocks when the variance of the implementation error increases. I show the contour plot at varying precision of central bank communication, assuming the implementation error of the interest rate has a standard deviation of 0.5. Since interest rate becomes a relatively imprecise signal now, the value of (external) information goes back to the conventional wisdom. The loss is minimized when information on natural rate shock is most precise and information on the cost push shock is least precise.
5 Dynamic Informational Effect

I extend the analysis to the dynamic informational effect of the interest rate by introducing serially correlated shocks. Since the consumption and pricing decisions are both forward-looking, the expectations on the future states matter for current output gap and inflation. When shocks have serial correlation, current interest rates affect expectations on future shocks as well, which leads to the dynamic informational effect of interest rates.

5.1 States, Beliefs and Equilibrium in Private Sector

To analyze the direct informational effect of interest rates, I first study the dynamic learning process in the private sector.

State
Natural-rate shocks and cost push-shocks follow AR(1) process:

\[
\begin{bmatrix}
    r_n^t \\
u_t
\end{bmatrix} = \begin{bmatrix}
    \phi & 0 \\
    0 & \phi^u
\end{bmatrix} \begin{bmatrix}
    r_n^{t-1} \\
u_{t-1}
\end{bmatrix} + \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
v_f^t \\
u_{f}^u
\end{bmatrix}.
\]

(42)

Signals
The information set of the private sector includes values of all parameters and the entire history of interest rates upon \( t \). I first guess and then show that the equilibrium interest rate becomes a function of the state variables in period \( t \), which includes both the actual shocks at time \( t \) and beliefs in the period \( t - 1 \).

\[
i_t = F_1 r_n^t + F_2 E_{t-1}^s r_n^{t-1} + F_3 u_t + F_4 E_{t-1}^s u_{t-1}.
\]

(43)

The reason for the current interest rate to react on past beliefs are due to the Kalman Filtering process in the private sector, which optimally weighs the signals in the current period and the beliefs in the past period to form expectations on current states. Consequently, the current output gap and inflation become functions of past beliefs. Therefore, when a discretionary central bank sets current interest rate to minimize deviations of the current output gap and inflation, the interest rate in equilibrium also react to beliefs in the past period.

As the private agents have perfect memory of their beliefs in the past, they are able to distinguish the fraction of the interest rate which reacts to current shocks from the fraction of the interest rate which reacts to past beliefs. Let \( \hat{i}_t \) denote the fraction of \( i_t \) that reacts to current shocks, which follows:

\[
\hat{i}_t = i_t - F_3 E_{t-1}^s r_n^{t-1} - F_4 E_{t-1}^s u_{t-1} = F_1 r_n^t + F_3 u_t.
\]

(44)
Beliefs Formation

The private sector forms expectation on current states through Kalman filtering process. Denote the hidden state variables as

$$z_t = \Phi z_{t-1} + v_t \quad (45)$$

where $$z_t = [r^n_t, u_t]'$$, $$\Phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix}$$, and $$v_t = [v^n_t, v^u_t]'$$ with white noise of variance $$Q$$.

Denote the observable signal as:

$$s_t = Dz_t \quad (46)$$

where $$s_t = i_t$$, and $$D = [F_1, F_3]'$$.

The Kalman Filtering process makes beliefs on current state variables as the optimal combination of beliefs in last period and signals in current period:

$$E_s^t z_t = \Phi E_s^{t-1} z_{t-1} + K \left(s_t - D \Phi E_s^{t-1} z_{t-1}\right) \quad (47)$$

where the optimal weight, $$K$$, is determined by Ricatti Iteration:

$$K = PD'(DPD')^{-1} \quad (48)$$
$$P = \Phi \left(P - PD'(DPD')^{-1}DP\right) \Phi + Q \quad (49)$$

Solution in Private Sector under Arbitrary Policy Coefficients

The equilibrium in private sector is described by the system of equations summarizing private sector optimization decisions in aggregate variables (equations 10 and 12), shock evolution (equation 32), interest rate reaction function (equation 33), and beliefs updating process characterized in equation (37).

Since the equilibrium involves forward-looking variables, I solve it by undetermined coefficients. I first conjecture that $$\hat{y}_t$$ and $$\pi_t$$ are linear functions of state variables in period $$t$$, that is,

$$[r^n_t, u_t, E^s_{t-1} r^n_{t-1}, E^s_{t-1} u_{t-1}]$$

$$\begin{bmatrix} \hat{y}_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \end{bmatrix} \begin{bmatrix} r^n_t \\ E^s_{t-1} r^n_{t-1} \\ u_t \\ E^s_{t-1} u_{t-1} \end{bmatrix} \quad (50)$$

This conjecture allows for the expression of expected future equilibrium variables in terms of
the beliefs of current shocks, $E_t^s r_t^u$ and $E_t^s u_t$:

$$
\begin{bmatrix}
E_t \hat{y}_{t+1}
E_t \pi_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\gamma_1 \phi + \gamma_2 & \gamma_3 \phi^u + \gamma_4 & E_t^s r_t^u
\gamma_5 \phi + \gamma_6 & \gamma_7 \phi^u + \gamma_8 & E_t^s u_t
\end{bmatrix}
$$

(51)

Substituting these into the IS and the Phillips curve results in the expression of $\hat{y}_t$ and $\pi_t$ as functions of actual shocks $[r_t^u, u_t]$ and beliefs $[E_t^s r_t^u, E_t^s u_t]$. Applying the beliefs updating process yields the expressions as functions that consist only predetermined states. (see Appendix B for detailed derivation)

### 5.2 Discretionary Monetary Policy

A discretionary central bank minimizes the expected output gap and inflation deviations in all periods. The central bank’s optimization problem can be written as:

$$
E_t L(t) = E_t [\pi_t^2 + \omega \hat{y}_t^2] + \beta E_t (L(t + 1))
$$

(52)

where the output gap follows equation (10), the inflation follows equation (12), the actual shocks evolve as equation (35), and beliefs are formed using Kalman filtering process as specified in equation (37 - 39).

$E_t$ denotes the objective expectation. The information set of the central bank at $t$ includes the entire history of natural rate and cost push shock upon $t$, and the beliefs formed in the private sector upon $t - 1$, i.e.,

$$
I_t = \{r_T^u, E_{T-1}^s r_{T-1}^u u_T, E_{T-1}^s u_{T-1}, \forall T = 0...t\}
$$

$E_t (L(t + 1))$ includes the deviations of equilibrium inflation and output gap in all future periods:

$$
E_t (L(t_1)) = \sum_{j=1}^{\infty} \beta^j E_t \begin{bmatrix}
\pi_{t+1} & \hat{y}_{t+j}
\end{bmatrix} \begin{bmatrix}
1 & 0
0 & \omega
\end{bmatrix} \begin{bmatrix}
\pi_{t+j}
\hat{y}_{t+j}
\end{bmatrix}
$$

(53)

When there is serial correlation in shocks, interest rate has dynamic information effect due to the persistent learning process in the private sector. Consequently, this dynamic information effect changes the objective function of a discretionary central bank.  

---

16As long as there are shocks of which central bank is unable to completely offset, optimal policy can be described as "leaning against wind" - seeking contemporary negative correlation between output gap and inflation. For discussion about the conventional within-period "leaning against" policy that is caused by the information frictions, see Adam (2005), Angeletos and La’O (2013), and Tang (2015), among others.
Lemma 5 With dynamic information effect, the optimizing discretionary monetary policy is dynamically "leaning against the wind" as it targets a negative correlation between current and future deviations of output gap and inflation.

This can be shown as the first order condition of the central bank objective function:

$$\left\{ \frac{\partial E_t \pi_t}{\partial i_t^*} E_t \pi_t + \omega \frac{\partial E_t \hat{y}_t}{\partial i_t^*} E_t \hat{y}_t \right\} = -\frac{1}{2} \sum_{j=1}^{\infty} \beta^j \left\{ \frac{\partial E_t \pi_{t+j}}{\partial i_t^*} E_t \pi_{t+j} + \omega \frac{\partial E_t \hat{y}_{t+j}}{\partial i_t^*} E_t \hat{y}_{t+j} \right\} \quad (54)$$

To see that the right-hand side is non-zero, we need to first specify how future equilibrium is affected by current beliefs, and how the current interest rate affects current beliefs. Denote the predetermined state variables at \( t \) as: \( z_t = [r_t^n, E_{t-1}^s r_{t-1}^n, u_t, E_{t-1}^s u_{t-1}] \). Due to the projected linear relation, the objective expectation of the inflation and the output gap in \( j \) period ahead becomes:

$$E_t z_{t+j} = \left[ \begin{array}{c} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{array} \right] = \left[ \begin{array}{cccc} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \end{array} \right] E_t z_{t+j} \equiv \Gamma E_t z_{t+j} \quad (55)$$

As long as shocks cannot be completely offset by the interest rate, \( \Gamma \) is non-zero.

The evolution of \( E_t z_{t+j} \) includes the auto-correlated actual shocks, and the dynamic process of beliefs formation. The beliefs formation yields:

$$E_t^s r_t^n = K_{11} F_1 r_t^n + \phi (1 - K_{11} F_1) E_{t-1}^s r_{t-1}^n + K_{11} F_3 u_t - \phi^u K_{11} F_3 E_{t-1}^s u_{t-1} \quad (56)$$

$$E_t^s u_t = K_{21} F_1 r_t^n - \phi K_{21} F_1 E_{t-1}^s r_{t-1}^n + K_{21} F_3 u_t + \phi^u (1 - K_{21} F_3) E_{t-1}^s u_{t-1} \quad (57)$$

Thus, the evolution of \( E_t z_{t+j} \) can be summarized as:

$$\left[ \begin{array}{c} E_t r_{t+j}^n \\ E_t E_{t+j-1}^s r_{t+j-1} \\ E_t u_{t+j} \\ E_t E_{t+j-1}^s u_{t+j-1} \end{array} \right] = \left[ \begin{array}{cccc} \phi & 0 & 0 & 0 \\ K_{11} F_1 & \phi - K_{11} F_1 \phi & K_{11} F_3 & -K_{11} F_3 \phi^u \\ 0 & 0 & \phi^u & 0 \\ K_{21} F_1 & -K_{21} F_1 \phi & K_{21} F_3 & \phi^u - K_{21} F_3 \phi^u \end{array} \right] \left[ \begin{array}{c} E_t r_{t+j-1}^n \\ E_t E_{t+j-2}^s r_{t+j-2} \\ E_t u_{t+j-1} \\ E_t E_{t+j-2}^s u_{t+j-2} \end{array} \right] \equiv \Lambda E_t z_{t+j-1} \quad (58)$$

Combine the equation (45) and (48) to express the future equilibrium in terms of current beliefs as:

$$\left[ \begin{array}{c} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{array} \right] = \Gamma \Lambda^{j-1} E_t z_t \quad (59)$$

Substituting this expression to the central bank’s objective function transforms the objective function into a weighted sum of the current inflation, the current output gap and the persistent state variables which include current actual shocks and current beliefs. The first order condition on \( i_t^* \)
results in:

$$\left\{ \frac{\partial E_t \pi_t}{\partial i_t} E_t \pi_t + \omega \frac{\partial E_t \hat{y}_t}{\partial i_t} \right\} + \frac{1}{2} \sum_{j=1}^{\infty} \beta^j \Delta(j-1) = 0 \quad (60)$$

where $\Delta$ captures how current interest rate affects future deviations through its information effect on $[E_t^s r_t^n, E_t^s u_t]$. (See Appendix for derivations.)

**Proposition 3:** With serially correlated shocks, the interest rate in the current period affects future equilibrium through the dynamic informational effect. The consideration of the dynamic informational effect makes the equilibrium interest rate to target beliefs in addition to targeting the current inflation and output gap.

The consideration of the dynamic informational effect consists two parts. The first part is captured in the effect on current equilibrium, because both consumption and pricing decisions are forward looking. The second part is due to the persistent in the learning process, which is in addition to stabilizing current economy. This additional beliefs-targeting does not exist with serially uncorrelated shocks.

The effects of discretionary policy on future variables are different from the effects on the actual future variables, as the private sector cannot distinguish the actual shocks from the beliefs.

To see this, use output gap as an example. First express future output gap as the actual shocks, the expected shocks and the interest rate.

$$\hat{y}_{t+1} = \Xi(1) E_{t+1}^s r_{t+1}^n + \Xi(2) E_{t+1}^s u_{t+1} + \frac{1}{\sigma} \frac{1}{1 - \phi} r_{t+1}^n - \frac{1}{\sigma} i_{t+1} \quad (61)$$

Next, express the expected shocks as the beliefs formed with weight on both past beliefs and signals in this period. 17

$$E_{t+1}^s r_{t+1}^n = \Lambda_1 E_t^s r_t^n + \Lambda_2 E_t^s u_t + K_{11} i_{t+1}$$

$$E_{t+1}^s u_{t+1} = \Lambda_3 E_t^s r_t^n + \Lambda_4 E_t^s u_t + K_{21} i_{t+1}$$

The marginal effect of the interest rate on $\hat{y}_{t+1}$ can then be expressed as the combination of the informational effect on $E_t^s r_t^n$ and $E_t^s u_t$.

$$\frac{\partial \hat{y}_{t+1}}{\partial i_t} = \Xi(1) \left( \Lambda_1 \frac{\partial E_t^s r_t^n}{\partial i_t} + \Lambda_2 \frac{\partial E_t^s u_t}{\partial i_t} \right) + \Xi(2) \left( \Lambda_1 \frac{\partial E_t^s r_t^n}{\partial i_t} + \Lambda_2 \frac{\partial E_t^s u_t}{\partial i_t} \right) \quad (62)$$

However, the effect of interest rate on the expected future output gap has the additional term, as the private sector is not able to separate the beliefs from the actual $r_{t+1}$. The effect of discretionary

17 See Appendix for expressions of $\Xi$ and $\Lambda$
policy on the expected output gap is:

\[
\frac{\partial E_t^s \hat{y}_{t+1}}{\partial i_t} = \left( Z(1) + \frac{1}{\sigma} \frac{1}{1 - \phi} \right) \left( \Lambda_1 \frac{\partial E_t^s r_t^n}{\partial i_t} + \Lambda_2 \frac{\partial E_t^s u_t}{\partial i_t} \right) + Z(2) \left( \Lambda_1 \frac{\partial E_t^s r_t^n}{\partial i_t} + \Lambda_2 \frac{\partial E_t^s u_t}{\partial i_t} \right)
\] (63)

In addition, the effect of discretionary policy on future variables should be distinguished from the effect of committed future interest rate, as the first one consists the information effect on the current beliefs, and the later one consists only the direct effect on future borrowing costs. Both of the two effects are able to influence current equilibrium when private agents are forward-looking. The marginal effect of an increase in the future interest rate on \( \hat{y}_{t+1} \) is:

\[
\frac{\partial \hat{y}_{t+1}}{\partial i_{t+1}} = -\frac{1}{\sigma}.
\] (64)

To solve for the equilibrium interest rate under discretion, I first propose that the interest rate follows a linear function,

\[
i_t = F_1 r_t^n + F_2 E_{t-1}^s r_{t-1}^n + F_3 u_t + F_4 E_{t-1}^s u_{t-1},
\]

with which the private sector updates beliefs on \( E_t^s r_t^n \) and \( E_t^s u_t \). The central bank then chooses the interest rate to minimize the loss function specified in equation (45). If the optimizing interest rate is different from the proposed one, the private sector then changes beliefs on the interest rate reaction function. Optimal interest rate is found as the fixed-point solution in this iteration process. Details of this solution method is provided in Appendix B.

The persistence in underlying shocks strengthens the information effect of interest rate, because it increases the effect of expected future deviations on current consumption and pricing decisions. If the serial correlation is high enough, it may cause optimal discretionary interest rate to fail to exist. The intuition is the following: suppose the private sector believes the best response of central bank is to increase the interest rate to the two shocks. If cost push shock realizes to be positive, which causes the inflation to positively deviate from steady state, the nominal effect of interest rate decreases inflation and the information effect of interest rate increases inflation. If the information effect dominates the direct effect, inflation increases even further. As a result, discretionary central bank wants to choose a negative interest rate, which contradicts the beliefs in the private sector that the best response of interest rate reacts positively to the two shocks.

### 5.3 Monetary Policy Rule

The objective function for commitment policy maker is the same as discretionary policy maker. I also restrict that central bank can only commits to a rule which responds linearly to current state variables. Notice that as past beliefs determine current beliefs, they also become current state variables. In equilibrium, the optimal follows same functional form as discretionary interest rate,
i.e., \( i_t = F_1 r_t^n + F_2 E_{t-1}^s r_{t-1}^n + F_3 u_t + F_4 E_{t-1}^s u_{t-1} \). The optimal rule chooses \([F_1, F_2, F_3, F_4]\) in order to minimize the ex-ante loss from the steady state. \(^{18}\)

\[
\min_{F_1, F_2, F_3, F_4} E_t L(t) = \int \int \left( \pi_t^2 + \omega \gamma_t^2 + \beta E_t L(t + 1) \right) dr_t^n du_t \tag{65}
\]

where output gap follows equation (10), inflation follows equation (12), actual shocks evolve as equation (34), and beliefs are formed using Kalman filtering process as specified in equation (39 - 41).

Different from the serially uncorrelated case, where I completely shut down the gains from commitment through delayed response, I also allow for such gains in the dynamic case. Potentially, policy rule can react to current cost-push shocks by a smaller amount, and commits on a large response to past beliefs than discretionary interest rate does. In doing so, not only does interest rate reveals less information about cost-push shock, it also decreases expected inflation. The gains from committing to delayed response strengthens the gains from informational effect.

6 Quantitative Assessment

The goal of this section is to quantify the size of gains from commitment using a dynamic model with varying degree of information precision. I start with the case where there are no external signals. In this case, the information precision depends on the prior distribution of the actual shocks. I then consider the case where there are external signals, and in addition, I allow for an implementation error in the interest rate. By varying the precision of signals, I quantitatively assess how the gains of commitment depend on the precision of external information.

6.1 No External Information

In the baseline model, I assume that there are no external signals, and calibrate model parameters in line with the convention of the macroeconomics literature. As noted in Section 2, I set \( \phi = 1 \) and \( \sigma = 1 \), assuming unitary Frisch elasticity of labor supply and log utility of consumption. I use \( \beta = 0.99 \), which implies a steady state real return on financial assets of four percent. For price rigidity, I calibrate \( \theta \), the price stickiness parameter to be 0.5, as the average price duration from macro empirical evidence and micro empirical evidences. \(^{19}\) For the parameter that governs the elasticity of substitution between intermediate goods, I set \( \epsilon = 4 \), which implies a steady state price markup to be one third of revenue.

\(^{18}\)In steady state, \( E_{t-1}^s r_{t-1}^n = 0 \), and \( E_{t-1}^s u_{t-1} = 0 \).

\(^{19}\)Sources:Bils and Klenow (2004), Gali and Gertler (1999), Nakamura and Steinsson (2010)
For the evolution of underlying shocks, I set the auto-correlation of natural rate shock to be 0.9, and standard deviation to be 3 percent which is measured by Laubach and Williams (2003). There are less consensus in the persistence and volatility of the cost-push shock, as it stems from a various sources. I set the auto-correlation for cost push shock to be 0.3 in order to avoid informational effect of interest rate to be so high that kills the equilibrium of optimizing discretionary interest rate. The standard deviation to be the same as that of natural rate shock. In addition, I set the standard deviation of policy implementation error to be the same as the standard deviation of natural rate shock. I assume that there are no external signals apart from the interest rate. Summary of parameter values in baseline calibration is provided in Appendix.

I numerically solved for both the equilibrium interest rate under discretion and the optimal policy rule under the baseline dynamic model, which results in:

\[
i_{\text{discretionary}} = 1.3887r^t_n - 0.3498E_{t-1}r^t_n + 0.1852u_t + 0.3370E_{t-1}u_{t-1}
\]
\[
i_{\text{rule}} = 1.3727r^t_n - 0.3374E_{t-1}r^t_n + 0.1830u_t + 0.3332E_{t-1}u_{t-1}
\]

For the equilibrium interest rate under discretion, what is new for the dynamic case is the value of \(F_2\) and \(F_4\). They capture how the interest rate optimally responses to the beliefs in the past period. As beliefs are persistent, reacting to past beliefs leads to inertia in the interest rate. As analyzed in the beliefs formation process, this response has not informational effect, as the private sector is able to subtract the part of \(F_2E_{t-1}r^t_n + F_4E_{t-1}u_{t-1}\) to get signal in current period. Specifically, a negative \(F_2\) means central bank counter-reacts the excess response to natural rate shock in the first period. A positive \(F_4\) means central bank make up for the deficient response to cost push shock in the first period. The intuition can be found in the output gap equation and inflation equation, which show the expected natural rate decreases output gap and expected cost push shock increases inflation. As past beliefs contribute positively to current beliefs, interest rate optimally responds to past beliefs in order to offsets their contribution to current deviations.

Comparing with the optimal response of a discretionary central bank, the response coefficients for optimal policy rule do not differ very much. This suggests that the welfare gains from commitment is not significant. I calculate the ex-ante welfare loss as the loss function of the central bank, which is 0.0325 for the discretionary case and 0.0324 for the commitment case. In Figure 8, I plot the impulse response after natural rate shock, cost push shock and policy error, which also show that equilibrium under discretionary policy and policy rule is negligible when interest rate is the only source of information.
6.2 Varying Precision of External Information

As analyzed in Section 4, the precision of external signals crucially determines the gains from commitment, as it affects the size of the informational effect of interest rate. However, the previous literature does not provide consensus on the degree of information frictions in the private sector. Instead of calibrating the precision of external signals, I investigate how the size of gains vary...
with the precision of external signals. I first set the variance of the external signals to be same as the variance of the ex-ante dispersion of the actual shocks. In addition, I also allow for an implementation error for interest rate, and set the standard deviation of the implementation error to be 0.01.

Figure 9 compares the impulse response function after natural rate shock, cost push shock and policy rate shock under discretionary policy and under policy rule.

Figure 9: Impulse Response under Discretionary Policy (Left) and Optimal Policy Rule (Right), with No External Information
In Figure 9, the interest rate is numerically solved as:

\[ i_{\text{discretionary}} = 1.4402r^p_t - 0.3962E^s_{t-1}r^p_{t-1} + 0.4485u_t + 0.2578E_{t-1}u_{t-1}, \]
\[ i_{\text{rule}} = 1.2818r^p_t - 0.2430E^s_{t-1}r^p_{t-1} + 0.0373u_t + 0.5184E_{t-1}u_{t-1}. \]

The most significant difference is that the interest rate under commitment responds much less to cost push shock, as \( F_{\text{comm}} = 0.0373 \) and in comparison, \( F_{\text{disc}} = 0.4485 \). At the same time, the conflict between output gap stabilization and inflation stabilization in the period where cost push shock occurs is largely reduced. There are two reasons why optimal policy rule is able to largely reduce stabilization bias. First, as interest rate commits to react less to cost push in current period, it reduces information revealed on cost push shock, which decreases the sensitivity of beliefs to interest rate. Second, as interest rate commits to react more in future period, this commitment of tightening policy further reduces expected inflation and expected output gap in future periods.

In the rest of this section, I show how the size of gains depends on the precision of external signals. First of all, holding the standard deviation of \( m^u_t \) fixed at 0.1, I vary the precision of external signal on natural rate shock from 0.01 to 0.1. \(^20\) In the following table I report the welfare gains measured by the ex-ante loss function of the central bank \(^21\) standard deviation of inflation and standard deviation of output gap for both types of central bank. When signal on natural rate changes from relatively imprecise \( \sigma_{ER} = 0.1 \) to very precise \( \sigma_{ER} = 0.01 \), the welfare gains measured by the ex-ante loss function of the central bank increases from 64 percent to 70.4 percent.

<table>
<thead>
<tr>
<th></th>
<th>Discretionary</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_{ER} = 0.01 )</td>
<td>5.13</td>
<td>1.84</td>
</tr>
<tr>
<td>( \sigma_{ER} = 0.05 )</td>
<td>3.78</td>
<td>1.89</td>
</tr>
<tr>
<td>( \sigma_{ER} = 0.1 )</td>
<td>3.76</td>
<td>1.12</td>
</tr>
<tr>
<td>Inflation</td>
<td>1.45</td>
<td>1.10</td>
</tr>
<tr>
<td>Output Gap</td>
<td>4.63</td>
<td>2.31</td>
</tr>
<tr>
<td></td>
<td>4.42</td>
<td>2.33</td>
</tr>
</tbody>
</table>

Table 1: Gains from Commitment at Varying Precision of External Information on \( r^p_t \)

The ex-ante loss is calculated as the objective function of the central bank defined in equation (57) \( \times 10^2 \). The numbers for inflation and output gap are noted in percentage points.

In the following table, I report the size of gains from commitment when holding the standard deviation of \( m^u_t \) fixed at 0.1, and varying the standard deviation of external signal on cost-push shock from 0.01 to 0.1. When signal on natural rate changes from relatively imprecise \( \sigma_{ER} = 0.1 \) to very precise \( \sigma_{ER} = 0.01 \), the welfare gains measured by the ex-ante loss function of the central bank increases from 64 percent to 70.4 percent.

\(^{20}\) Standard deviation of interest rate implementation error is fixed to be 0.01 in all calibration exercises.

\(^{21}\) This is defined as the objective function for commitment central bank in equation (57), which is the weighted sum of squared deviations from steady state.
bank increases from 28 percent to 49 percent.

<table>
<thead>
<tr>
<th></th>
<th>Discretionary</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{\epsilon} = 0.01$</td>
<td>$\sigma_{\epsilon} = 0.05$</td>
</tr>
<tr>
<td>Ex-ante Loss</td>
<td>6.28</td>
<td>3.76</td>
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<tr>
<td>Inflation</td>
<td>1.39</td>
<td>1.12</td>
</tr>
<tr>
<td>Output Gap</td>
<td>5.99</td>
<td>4.56</td>
</tr>
</tbody>
</table>

Table 2: Gains from Commitment at Varying Precision of External Information on $u_t$

The ex-ante loss is calculated as the objective function of the central bank defined in equation (57) $\times 10^2$. The numbers for inflation and output gap are noted in percentage points.

7 Conclusion

In this paper, I studied an economy where the private sector has imperfect information about the underlying shocks and the central bank has perfect information when making interest-rate decisions. Consequently, the interest rate decisions have an informational effect as the private sector regards the equilibrium interest rate to be a signal about the unobserved shocks, and extracts information from the interest rate. I showed that with serially correlated shocks and relatively precise external information, the size of gains is quantitatively important.

To theoretically study the gains from commitment, I built a New Keynesian model with both nominal frictions and information frictions, and studied the optimal response of interest rates to natural-rate shocks and cost-push shocks. I started with the simple scenario where both shocks are serially uncorrelated, which allowed me to isolate the informational gains from commitment on beliefs in the current period. Using an arbitrary interest rate function that responds positively to both shocks, I showed that beliefs in the private sector are more sensitive to the shock which interest rate reacts more aggressively to or has higher ex-ante dispersion.

I began the analysis of monetary policy by characterizing how the informational effect of interest rates changes the Phillips curve, which is the constraint faced by the central bank. As the Phillips curve measures the co-movement of the output gap and the inflation that results from changes in the interest rate, it is affected by the informational effect of the interest rate, and is further determined by how private agents expect interest rate to respond to different shocks.

A discretionary central bank sets interest rates to optimize its objective function at any state of the economy, taking as given the informational effect of its interest-rate decision. In comparison, under commitment, a central bank can change the informational effect of interest rates by committing to a different response function than its optimizing response under discretion. Consequently, the informational effect of interest rates makes the Phillips curve endogenous to the central bank’s
decision about the optimal policy rule.

The response of optimal policy rule to the two shocks are jointly determined. Assuming that
the natural-rate shock and the cost-push shock have same ex-ante dispersion, I show that optimal
policy rule responds more aggressively to the natural-rate shock and less aggressively to the cost-push shock, relative to the equilibrium optimizing interest rate set by discretionary central bank. By
doing so, it achieves an informational advantage as it withholds information about the cost-push shock, which consequently reduces the stabilization bias caused by the actual cost-push shocks under perfect information.

I extend the analysis by studying the interaction between the informational effect of interest rate and external signals. Central bank direct communication can also be modeled as providing more precise external signals independent to the informational effect of interest rates. I showed the situation where providing more precise information about the efficient shocks might reduce welfare. This happens when communicating about the natural-rate shock also makes the interest-rate a more precise signal about the cost-push shock.

Lastly, I quantified the size of gains from commitment by adopting conventionally used parameter values with varying precision of external signals. I found that when external signals are extremely imprecise, the size of gains from commitment is negligible. On the other hand, more precise external information on both shocks increases the size of gains from commitment. Specifically, when the precision of external signals is equal to the prior distribution of actual shocks, committing to the optimal policy rule improves ex-ante welfare by 54 percent relative to the equilibrium under optimizing discretionary policy.
References


Appendices

A Log-Linearization and Aggregation

From the household first order conditions, we first do log-linear approximation to the Euler equation in (A.6) by

\[ y_t = E_t y_{t+1} - \frac{1}{\sigma} \left( i_t - E_t \pi_{t+1} \right) \]  

(A.1)

The log-linear approximation to the labor supply of equation (A.7) is

\[ \phi n_t(j) + \sigma y_t = w_t(j) \]

where \( w_t \) denotes the log approximated real wage, \( \log(W_t/P_t) \). Recall that resource constraint implies that \( c^j_t = y^j_t \) \( \forall j \), which further implies \( c_t = y_t \). We can then write the labor supply as follows:

\[ \phi n_t(j) + \sigma y_t = w_t(j) \]  

(A.2)

Next, we want to relate individual firm’s real marginal cost of production to aggregate output. To do this, first integrate equation (A.13):

\[ \int w_t(j) = \phi \int n_t(j) dj + \sigma y_t \]  

(A.3)

Then, substitute the log-linear approximation of the individual good demand, i.e., \( y_t(j) - y_t = -\epsilon(p_t(j) - p_t) \), which results in:

\[ \int n_t(j) dj = y_t + \int (-\epsilon)(p_t(j) - p_t) - \int a_t(j) = y_t - a_t \]  

(A.4)

Substitute this into \( \int w_t(j) \), and then deduct \( a_t \) from both sides:

\[ \int w_t(j) - a_t(j) = (\phi + \sigma)y_t - (1 + \phi)a_t \]  

(A.5)

Define natural level of output as the equilibrium output level without price rigidity and under perfect information, which makes \( y^n_t \) as a linear function of aggregate technology. Then, write the above equation in terms of output gap:

\[ \int w_t(j) - a_t(j) = (\phi + \sigma)(y_t - y^n_t) \]  

(A.6)

We know move on to the firm’s side. Taking log-linear approximation of individual firm’s optimal resetting prices:

\[ p^*_t(j) = (1 - \beta \theta)E_t^j \left\{ \Sigma \beta^j \left[ p_{t+k} + u_{t+k}(j) + w_{t+k}(j) - a_{t+k}(j) \right] \right\} \]  

(A.7)
The Calvo assumption implies that the aggregate price index is an average of the price charged by the fraction of $1 - \theta$ of firms which reset their prices at $t$, and the fraction of $\theta$ of firms whose prices remain as the last period prices. Thus, the log-linear approximation of the aggregate price in period $t$ becomes:

$$p_t = \theta p_{t-1} + (1 - \theta) \int p_t^*(j) d j$$  \hspace{1cm} (A.8)

Subtract $p_{t-1}$ from both sides to express in terms of inflation:

$$\pi_t = (1 - \theta) \left( \int p_t^*(j) - p_{t-1} \right)$$  \hspace{1cm} (A.9)

As explained in Section 2.3, assume homogeneous, subjective believes in order to abstract from the higher order beliefs problem in aggregating prices. This assumption allows me to write individual resetting prices as:

$$p_t^*(j) = (1 - \beta \theta)(E_t^s p_t + u_t(j) + w_t(j) - a_t(j)) + (1 - \beta \theta)\sum_{k=1}^{\infty} (\beta \theta)^k E_t^s (p_{t+k} + u_{t+k}(j) + w_{t+k} - a_{t+k})$$  \hspace{1cm} (A.10)

Integrate over $j$:

$$\int p_t^*(j) d j = (1 - \beta \theta)(E_t^s p_t + u_t + w_t - a_t) + (1 - \beta \theta)\sum_{k=1}^{\infty} (\beta \theta)^k E_t^s (p_{t+k} + w_{t+k} - a_{t+k})$$  \hspace{1cm} (A.11)

To write in difference equation, first calculate:

$$\beta \theta \int E_t^s p_{t+1}^*(j) d j = (1 - \beta \theta)\sum_{k=1}^{\infty} E_t^s (p_{t+k} + w_{t+k} - a_{t+k}) = \beta \theta E_t^s p_{t+1}^*$$  \hspace{1cm} (A.12)

The second equation holds due to homogeneous beliefs.

Subtract equation (A. 23) from equation (A. 22):

$$\int p_t^*(j) d j - \beta \theta E_t^s p_{t+1} = (1 - \beta \theta)E_t^s p_t + (1 - \beta \theta)u_t + (1 - \beta \theta)(\varphi + \sigma)\hat{\gamma}_t$$  \hspace{1cm} (A.13)

$$\int p_t^*(j) d j - p_{t-1} = \beta \theta (E_t^s p_{t+1}^* - E_t^s p_t) + E_t^s p_t - p_{t-1} + (1 - \beta \theta)u_t + (1 - \beta \theta)(\varphi + \sigma)\hat{\gamma}_t$$

$$\pi_t = \beta \theta E_t^s \pi_{t+1} + (1 - \theta)E_t^s \pi_t + (1 - \theta)(1 - \beta \theta)u_t + (1 - \beta \theta)(1 - \theta)(\varphi + \sigma)\hat{\gamma}_t$$

In the last equation, I assume that aggregate price is observable after one period, i.e., $p_{t-1} = E_t^s p_{t-1}$

Write inflation as:

$$\pi_t = \beta \theta E_t^s \pi_{t+1} + (1 - \theta)E_t^s \pi_t + k \theta \hat{\gamma}_t + u_t$$  \hspace{1cm} (A.14)

where $k = \frac{(1 - \beta \theta)(1 - \theta)(\varphi + \sigma)}{\theta}$, and $u_t = (1 - \theta)(1 - \beta \theta)u_t$
B Solution to the Markov Perfect Equilibrium under Discretionary Monetary Policy

In this section, I first solve the model with serially uncorrelated shocks and then solve the model with serially correlated shocks. For both cases, I solve for the fixed point where the beliefs by people in the private sector on the best response of interest rate at any state match the optimizing discretionary interest rate. This means that in equilibrium people have rational expectation.

B.1 Equilibrium Optimizing Discretionary Policy with Serially Uncorrelated Shocks

The solution takes the following steps:

1. I conjecture that interest rate reacts linear to both shocks, i.e., \( i_t = F^0_r r^n_t + F^0_u u_t \).

2. With this interest rate, I solve for the beliefs formed about natural-rate shock and cost-push shock in the private sector as functions of interest rate.

3. With beliefs formed in private sector, \( E^s_t r^n_t \) and \( E^s_t u_t \), the actual shocks, \( r^n_t \) and \( u_t \), I solve for \( \hat{y}_t \) and \( \pi_t \) as a function of \( i_t \).

4. Solve for \( i_t \) that minimizes the loss function, \( L_t = \pi_t^2 + \omega \hat{y}_t \), and express interest rate as actual shocks, \( i_t = F_r r^n_t + F_u u_t \).

5. Iterate the process until convergence.

Specifically, in step 1, \( i_t = F^0_r r^n_t + F^0_u u_t \).

In step 2, beliefs about underlying shocks follow:

\[
E^s_t r^n_t = K_r i_t \tag{B.1}
\]

\[
E^s_t u_t = K_u i_t \tag{B.2}
\]

where \( K_r F^0_r = \frac{F^0_r \sigma_r^2}{F^0_r \sigma_r^2 + F^0_u \sigma_u^2} \), and \( K_u F^0_u = \frac{F^0_u \sigma_u^2}{F^0_r \sigma_r^2 + F^0_u \sigma_u^2} \).

In step 3, write out the expression of output gap and inflation as function of interest rate:

\[
\hat{y}_t = -\frac{1}{\sigma} (i_t - r^n_t) \tag{B.3}
\]

\[
\pi_t = \kappa \hat{y}_t + (1 - \theta) \frac{\kappa}{\sigma} (E^s_t r^n_t(i_t) - r^n_t) + \frac{1 - \theta}{\theta} E^s_t u_t(i_t) + u_t \tag{B.4}
\]

In step 4, I first write out the first order condition of interest rate:

\[
\pi_t \frac{\partial \pi_t}{\partial i_t} + \omega \hat{y}_t \frac{\partial \hat{y}_t}{\partial i_t} = 0 \tag{B.5}
\]
Substitute \( \pi_t \) and \( \hat{y}_t \) by equation (B.3) and (B.4):

\[
\left\{ (1 - \theta) \frac{\kappa}{\sigma} (E^s_i r^n_i - r^n_i) + \frac{1 - \theta}{\theta} E^s_i u_t + u_t \right\} \frac{\partial \pi_t}{\partial i_t} + \left( \omega \frac{\partial \hat{y}_t}{\partial i_t} + \kappa \frac{\partial \pi_t}{\partial i_t} \right) \left\{ -\frac{1}{\sigma} (i_t - r^n_t) \right\} = 0 \quad (B.6)
\]

Substituting \( E^s_i r^n_i \) and \( E^s_i u_t \) as \( i_t \) leads to:

\[
\lambda_1 r^n_i + \lambda_2 u_t + \lambda_3 i_t = 0 \quad (B.7)
\]

where \( \frac{\partial \hat{y}_t}{\partial i_t} = -\frac{1}{\sigma} \), and \( \frac{\partial \pi_t}{\partial u_t} = -\frac{\kappa}{\sigma} + (1 - \theta) \frac{\kappa}{\sigma} K_r + \frac{1 - \theta}{\theta} K_u \), and

\[
\lambda_1 = \left\{ \left( \frac{\kappa}{\sigma} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma} - \frac{\partial \pi_t}{\partial i_t} (1 - \theta) \right\}
\]

\[
\lambda_2 = \frac{\partial \pi_t}{\partial i_t}
\]

\[
\lambda_3 = \frac{\partial \pi_t}{\partial i_t} (1 - \theta) \frac{\kappa}{\sigma} K_{11} + \frac{\partial \pi_t}{\partial i_t} (1 - \theta) K_{21} - \left( \frac{\kappa}{\sigma} \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma}
\]

Rearranging the above equation to get:

\[
i_t = F_1 r^n_i + F_3 u_t \quad (B.8)
\]

where \( F_1 = -\frac{\lambda_1}{\lambda_3} \), and \( F_3 = -\frac{\lambda_2}{\lambda_3} \).

In step 5, I iterate the above process until \( F_r = F_r^0 \) and \( F_u = F_u^0 \).

### B.2 Equilibrium Optimizing Discretionary Policy with Serially Correlated Shocks

In this section, I solve for the general version of the dynamic information case where I have serially correlated shocks, external signals which captures central bank direct communication, and implementation error.

The solution method is similar to the case with serially uncorrelated shocks, as solving for optimizing interest rate in equilibrium involves conjecture of interest rate response function. In addition to this conjecture, solving equilibrium variables in the private sector also requires additional step of undetermined coefficient to deal with the subjective expectation of future equilibrium variables.

1. I conjecture that interest rate reacts linear to both current shocks and past beliefs, i.e., \( i_t = F_1^0 r^n_i + F_2^0 E^s_i r^n_{i-1} + F_3^0 u_t + F_4^0 E^s_i u_{i-1} \).

2. With this interest rate, I solve for the beliefs formed about natural-rate shock and cost-push shock in the private sector as functions of current signals (interest rate and central bank...
communication) plus past beliefs.

3. (Undetermined Coefficient) I conjecture that output gap and inflation are linear functions of current state variables which include actual shocks and past beliefs. As a result, I am able to express the forward-looking output gap and inflation as functions of current actual shocks and current beliefs.

4. With beliefs formed in private sector, \( E_t^s r^n_t \) and \( E_t^s u_t \), the actual shocks, \( r^n_t \) and \( u_t \), I solve for \( \hat{y}_t \) and \( \pi_t \) as a function of \( i_t \).

5. Solve for \( i_t \) that minimizes the loss function, \( L_t = \pi_t^2 + \omega \hat{y}_t \), and express interest rate as actual shocks, \( i_t = F_1 r^n_t + F_3 u_t \).

6. Iterate the process until convergence.

Specifically, in step 1, I conjecture that

\[
i_t = F_3 E_t^{s-1} r^n_{t-1} + F_4 E_t^{s-1} u_{t-1}.
\]

In Step 2, to solve the beliefs formed in the private sector, I first specify the evolution of actual shocks:

**State:**

\[
\begin{bmatrix}
    r^n_t \\
    u_t
\end{bmatrix} =
\begin{bmatrix}
    \phi & 0 \\
    0 & \phi u_t
\end{bmatrix}
+ 
\begin{bmatrix}
    v_t \\
    v_u^t
\end{bmatrix}
\]

(B.9)

which I denote as \( z_t = \Phi z_{t-1} + v_t \), where \( \Phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \) and \( v_t = [v_t, v_u^t] \) with the white noise of variance \( Q \).

**Signals**

As people in private sector have perfect memory of beliefs they have in the past, they are able to back out the part of interest rate that reacts to current shocks, which I denote as

\[
\hat{i}_t \equiv i_t - F_3 E_t^{s-1} r^n_{t-1} - F_4 E_t^{s-1} u_{t-1}
\]

(B.10)

All signals are summarized as

\[
\begin{bmatrix}
    \hat{i}_t \\
    m^n_t \\
    m^u_t
\end{bmatrix} =
\begin{bmatrix}
    F_1 & F_3 \\
    1 & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    r^n_t \\
    u_t
\end{bmatrix}
+ 
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    e_t \\
    e^n_t \\
    e^u_t
\end{bmatrix}
\]

(B.11)

which I denote as \( s_t = Dz_t + R_t \)

**Beliefs**
People in private sector are Bayesian, and update beliefs through the Kalman Filtering process, in which they optimally weigh between all current signals and past beliefs by their variances. The beliefs follow:

\[
\begin{bmatrix}
E^s r^n_t \\
E^s u_t
\end{bmatrix} = \begin{bmatrix}
\phi & 0 \\
0 & \phi^u
\end{bmatrix} \begin{bmatrix}
E^s_{t-1} r^n_{t-1} \\
E^s_{t-1} u_{t-1}
\end{bmatrix} + \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23}
\end{bmatrix} \left( \begin{bmatrix}
\hat{\gamma}_t \\
\hat{\pi}_t
\end{bmatrix} - \begin{bmatrix}
F_1 \\
F_3
\end{bmatrix} + \begin{bmatrix}
\phi & 0 \\
0 & \phi^u
\end{bmatrix} \begin{bmatrix}
E^s_{t-1} r^n_{t-1} \\
E^s_{t-1} u_{t-1}
\end{bmatrix} \right)
\]

Write out the expression for \( \hat{\gamma}_t \) and collect terms:

\[
E^s r^n_t = (K_{11}F_1 + K_{12}) r^n_t + \phi (1 - K_{11}F_1 - K_{12}) E^s_{t-1} r^n_{t-1} + (K_{11}F_3 + K_{13}) u_t + \phi^u (-K_{11}F_3 - K_{13}) E^s_{t-1} u_{t-1} + K_{12} \epsilon_t^r + K_{13} \epsilon_t^u + K_{11} \epsilon_t
\]

\[
E^s u_t = (K_{21}F_1 + K_{22}) r^n_t + \phi (-K_{21}F_1 - K_{22}) E^s_{t-1} r^n_{t-1} + (K_{21}F_3 + K_{23}) u_t + \phi^u (1 - K_{21}F_3 - K_{23}) E^s_{t-1} u_{t-1} + K_{22} \epsilon_t^r + K_{23} \epsilon_t^u + K_{21} \epsilon_t
\]

Denote the above equations as \( E^s r^n_t = \Psi(1) r^n_t + \Psi(2) E^s_{t-1} r^n_{t-1} + \Psi(3) u_t + \Psi(4) E^s_{t-1} u_{t-1} + \Psi(5) \epsilon_t^r + \Psi(6) \epsilon_t^u + \Psi(7) \epsilon_t \), and \( E^s u_t = \Psi(8) r^n_t + \Psi(9) E^s_{t-1} r^n_{t-1} + \Psi(10) u_t + \Psi(11) E^s_{t-1} u_{t-1} + \Psi(12) \epsilon_t^r + \Psi(13) \epsilon_t^u + \Psi(14) \epsilon_t \). I will use this notation in solving equilibrium in the private sector by the method of undetermined coefficients.

In step 3, the first write out the the forward-looking output gap and inflation as:

\[
\hat{\gamma}_t = E^s r^n_t - \frac{1}{\sigma} \left[ \gamma_t - \left( \frac{1}{1 - \phi} r^n_t - \frac{\phi}{1 - \phi} E^s r^n_t \right) - E^s \pi_t \right]
\]

\[
\pi_t = \beta \theta E^s \pi_{t+1} + (1 - \theta) E^s \pi_t + \kappa \theta \hat{\gamma}_t + u_t
\]

Following the method of undetermined coefficients, I first need to conjecture that equilibrium variables are linear functions to current state variables, which include current actual shocks \((r^n_t, u_t)\), past beliefs, \(E^s_{t-1} r^n_{t-1}, E^s_{t-1} u_{t-1}\), and noise in current signals, \(\epsilon_t^r, \epsilon_t^u, \epsilon_t\).

\[
\begin{bmatrix}
\hat{\gamma}_t \\
\hat{\pi}_t
\end{bmatrix} = \begin{bmatrix}
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\
\gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11}
\end{bmatrix} \begin{bmatrix}
E^s_{t-1} r^n_{t-1} \\
E^s_{t-1} u_{t-1}
\end{bmatrix} + \begin{bmatrix}
\gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11}
\end{bmatrix} \begin{bmatrix}
\epsilon_t^r \\
\epsilon_t^u \\
\epsilon_t
\end{bmatrix}
\]

Next, substitute this conjecture into the forward-looking variables, \(E^s r^n_{t+1}\) and \(E^s \pi_{t+1}\). Notice
that noise of all signals are temporary, which are expected to be zero in future period.

\[
\begin{bmatrix}
E_t^s \hat{y}_{t+1} \\
E_t^s \pi_{t+1}
\end{bmatrix} = \begin{bmatrix}
\gamma_1 \phi + \gamma_2 & \gamma_3 \phi^u + \gamma_4 \\
\gamma_8 \phi + \gamma_9 & \gamma_{10} \phi^u + \gamma_{11}
\end{bmatrix} \begin{bmatrix}
E_t^s r_t^n \\
E_t^s u_t
\end{bmatrix}
\] (B.18)

First substitute this into the output gap expression:

\[
\hat{y}_t = \left( \gamma_1 \phi + \gamma_2 + \frac{1}{\sigma} (\gamma_8 \phi + \gamma_9) - \frac{1}{\sigma} \phi \right) E_t^s r_t^n + \left( \gamma_3 \phi^u + \gamma_4 + \frac{1}{\sigma} (\gamma_{10} \phi^u + \gamma_{11}) \right) E_t^s u_t - \frac{1}{\sigma} i_t + \frac{1}{\sigma} \phi r_t^n
\] (B.19)

Next work on \( \pi_t \), as the actual inflation also includes the expected current inflation, and expected current inflation includes expected current output gap, I first need to calculate:

\[
E_t^s \hat{y}_t = E_t \hat{y}_{t+1} - \frac{1}{\sigma} \left[ i_t - E_t^s r_t^n - E_t^s \pi_{t+1} \right]
\] (B.20)

\[
E_t^s \pi_t = \beta E_t^s \pi_{t+1} + \kappa \left\{ E_t^s \hat{y}_{t+1} - \frac{1}{\sigma} \left[ i_t - E_t^s r_t^n - E_t^s \pi_{t+1} \right] \right\} + \frac{1}{\theta} E_t^s u_t
\] (B.21)

Substitute \( E_t \pi_t \) into \( \pi_t \):

\[
\pi_t = \beta \theta E_t^s \pi_{t+1} + (1 - \theta) \left\{ \beta E_t^s \pi_{t+1} + \kappa E_t^s \hat{y}_t + \frac{1}{\theta} E_t^s u_t \right\} + \kappa \theta \hat{y}_t + u_t
\] (B.22)

\[
= \beta E_t^s \pi_{t+1} + (1 - \theta) \kappa \{ (\gamma_1 \phi + \gamma_2) E_t^s r_t^n + (\gamma_3 \phi^u + \gamma_4) E_t^s u_t \} - (1 - \theta) \frac{\kappa}{\sigma} i_t + (1 - \theta) \frac{\kappa}{\sigma} E_t^s r_t^n
\]

\[
+ (1 - \theta) \frac{\kappa}{\sigma} \{ (\gamma_8 \phi + \gamma_9) E_t^s r_t^n + (\gamma_{10} \phi^u + \gamma_{11}) E_t^s u_t \} + \frac{1 - \theta}{\theta} E_t^s u_t + \kappa \theta \hat{y}_t + u_t
\]

\[
= \left\{ (1 - \theta) \kappa (\gamma_1 \phi + \gamma_2) + (1 - \theta) \frac{\kappa}{\sigma} \right\} (\gamma_8 \phi + \gamma_9) E_t^s r_t^n
\]

\[
+ \left\{ (1 - \theta) \kappa (\gamma_3 \phi^u + \gamma_4) + \frac{1 - \theta}{\theta} + \left( \beta + (1 - \theta) \frac{\kappa}{\sigma} \right) (\gamma_{10} \phi^u + \gamma_{11}) \right\} E_t^s u_t - (1 - \theta) \frac{\kappa}{\sigma} i_t + \kappa \theta \hat{y}_t + u_t
\]

The values of \( \gamma \) can be solved in the following matrix:
\[ \gamma = M \gamma + c \]

where

\[ \gamma = [\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8]^T \]

\[ M = \begin{bmatrix}
\phi \psi_1 & \psi_1 & \phi^* \psi_8 & \psi_8 & 0 & 0 & 0 & \frac{1}{3} \phi \psi_1 \\
\phi \psi_2 & \psi_2 & \phi^* \psi_9 & \psi_9 & 0 & 0 & 0 & \frac{1}{3} \phi \psi_2 \\
\phi \psi_3 & \psi_3 & \phi^* \psi_{10} & \psi_{10} & 0 & 0 & 0 & \frac{1}{3} \phi \psi_3 \\
\phi \psi_4 & \psi_4 & \phi^* \psi_{11} & \psi_{11} & 0 & 0 & 0 & \frac{1}{3} \phi \psi_4 \\
\phi \psi_5 & \psi_5 & \phi^* \psi_{12} & \psi_{12} & 0 & 0 & 0 & \frac{1}{3} \phi \psi_5 \\
\phi \psi_6 & \psi_6 & \phi^* \psi_{13} & \psi_{13} & 0 & 0 & 0 & \frac{1}{3} \phi \psi_6 \\
\phi \psi_7 & \psi_7 & \phi^* \psi_{14} & \psi_{14} & 0 & 0 & 0 & \frac{1}{3} \phi \psi_7 \\
(1 - \theta) \phi \psi_1 + \kappa \theta & (1 - \theta) \phi \psi_2 & (1 - \theta) \phi \psi_3 + \kappa \theta & (1 - \theta) \phi \psi_4 & (1 - \theta) \phi \psi_5 & (1 - \theta) \phi \psi_6 + \kappa \theta & (1 - \theta) \phi \psi_7 & (1 - \theta) \phi \psi_8 + \kappa \theta & (1 - \theta) \phi \psi_9 & (1 - \theta) \phi \psi_{10} + \kappa \theta & (1 - \theta) \phi \psi_{11} + \kappa \theta & (1 - \theta) \phi \psi_{12} + \kappa \theta & (1 - \theta) \phi \psi_{13} + \kappa \theta & (1 - \theta) \phi \psi_{14} + \kappa \theta
\end{bmatrix} \]

\[ c = \begin{bmatrix}
-\frac{1}{3} \phi \psi_1 - \frac{1}{3} F_1 + \frac{1}{3} \phi \psi_1 \\
-\frac{1}{3} \phi \psi_2 - \frac{1}{3} F_2 \\
-\frac{1}{3} \phi \psi_3 - \frac{1}{3} F_3 \\
-\frac{1}{3} \phi \psi_4 - \frac{1}{3} F_4 \\
-\frac{1}{3} \phi \psi_5 \\
-\frac{1}{3} \phi \psi_6 \\
-\frac{1}{3} \phi \psi_7 - \frac{1}{3} F_7 \\
(1 - \theta) \phi \psi_1 + \frac{1}{3} \phi \psi_8 - (1 - \theta) \phi \psi_1 \\
(1 - \theta) \phi \psi_2 + \frac{1}{3} \phi \psi_9 - (1 - \theta) \phi \psi_2 \\
(1 - \theta) \phi \psi_3 + \frac{1}{3} \phi \psi_{10} - (1 - \theta) \phi \psi_3 \\
(1 - \theta) \phi \psi_4 + \frac{1}{3} \phi \psi_{11} - (1 - \theta) \phi \psi_4 \\
(1 - \theta) \phi \psi_5 + \frac{1}{3} \phi \psi_{12} - (1 - \theta) \phi \psi_5 \\
(1 - \theta) \phi \psi_6 + \frac{1}{3} \phi \psi_{13} - (1 - \theta) \phi \psi_6 \\
(1 - \theta) \phi \psi_7 + \frac{1}{3} \phi \psi_{14} - (1 - \theta) \phi \psi_7 \\
\end{bmatrix} \]

\( \gamma \) can be uniquely pinned down by the above linear system.
In step 5, in order to solve for the optimizing interest rate, I first need to specify central bank’s objective function.

**Central Bank Objective Function**

As current interest rate has persistent effect through the dynamic learning process, central bank also considers how current interest rate affect future equilibrium. Consequently, the loss function includes output gap and inflation of current and all future periods.

\[ E_i L(t) = \left[ \pi_t^2 + \omega \hat{\pi}_t^2 \right] + \beta E_i (L(t + 1)) \]  

where the \( E_i (L(t + 1)) \) is:

\[ \sum_{j=1}^{\infty} \beta^j E_i \left\{ \begin{bmatrix} \pi_{t+j} \\ \hat{\pi}_{t+j} \end{bmatrix} \right\} = \sum_{j=1}^{\infty} \beta^j \left\{ \begin{bmatrix} E_i \pi_{t+j} \\ E_i \hat{\pi}_{t+j} \end{bmatrix} \right\} + \text{indept. terms} \]  

The central banks expectation is *objective*, denoted by \( E_i \), in the sense that it observes all past shocks, and expects all future shocks to be zero. The information set of central bank at period \( t \) is:

\[ I_t = \{ r_t^n, u_T; \forall T = 0...t \} \]

Let \( z_t = [r_t^n, E_{i-1} r_t^n, u_t, E_{i-1} u_{t-1}]' \) denote the persistent state variables. So the central bank’s objective expectation of future period output gap and inflation becomes a linear function of \( E_i \tilde{z}_{t+j} \):

\[ \begin{bmatrix} E_i \pi_{t+j} \\ E_i \hat{\pi}_{t+j} \end{bmatrix} = \begin{bmatrix} \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} \\ \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} E_i \tilde{z}_{t+j} \equiv \Gamma E_i \tilde{z}_{t+j} \]  

\( E_i \tilde{z}_{t+j} \) follows:

\[ \begin{bmatrix} E_i r_{t+j}^n \\ E_i s_{t+j-1}^t r_{t+j-1}^n \\ E_i u_t \\ E_i s_{t+j-1}^t u_{t+j-1} \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & 0 \\ K_{11} F_1 + K_{12} & \phi (1 - K_{11} F_1 - K_{12}) & K_{11} F_2 + K_{13} & -\phi' (K_{11} F_3 + K_{13}) \\ 0 & 0 & \phi' & 0 \\ K_{21} F_1 + K_{22} & -\phi (K_{21} F_1 + K_{22}) & K_{21} F_2 + K_{23} & \phi' (1 - K_{21} F_3 - K_{23}) \end{bmatrix} \begin{bmatrix} E_i r_{t+j-1}^n \\ E_i s_{t+j-2}^t r_{t+j-2}^n \\ E_i u_{t-1} \\ E_i s_{t+j-2}^t u_{t+j-2} \end{bmatrix} \]  

(B.26)

Substitute into the \( E_i (L(t + 1)) \):

\[ \Sigma \beta^j E_i \tilde{z}_{t+1}^j (\Lambda^{j-1})' \Gamma' \Omega \Lambda^{j-1} E_i \tilde{z}_{t+1} \equiv \Sigma \beta^j E_i \tilde{z}_{t+1}^j \Theta_{j-1} E_i \tilde{z}_{t+1} \]  

(B.28)

Take the first order condition on \( i_t^n \) of \( E_i L(t + 1) \):

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\[
\left\{ \frac{\partial E_t \pi_t}{\partial i_t} E_t \pi_t + \omega \frac{\partial E_t \hat{y}_t}{\partial i_t} E_t \hat{y}_t \right\} + \frac{1}{2} \Sigma_{j=1}^{\infty} \beta^j \Delta(j - 1) = 0 \tag{B.29}
\]

where

\[
\Delta_{j-1} = (\Theta_{j-1}^{21} + \Theta_{j-1}^{12}) \phi r^n_{i_t} \frac{\partial E_t^s r^n_{i_t}}{\partial i_t} + (\Theta_{j-1}^{32} + \Theta_{j-1}^{23}) \phi^u u_t \frac{\partial E_t^s r^n_{i_t}}{\partial i_t} + (\Theta_{j-1}^{42} + \Theta_{j-1}^{34}) E_t^s u_t \frac{\partial E_t^s r^n_{i_t}}{\partial i_t} + \Theta_{j-1}^{22} \cdot 2 E_t^s r^n_{i_t} \frac{\partial E_t^s r^n_{i_t}}{\partial i_t}
\]

\[
\equiv \Delta_{j-1}(1) r^n_{i_t} + \Delta_{j-1}(2) u_t + \Delta_{j-1}(3) E_t^s u_t + \Delta_{j-1}(4) E_t^s r^n_{i_t} + \Delta_{j-1}(5) r^n_{i_t} + \Delta_{j-1}(6) u_t + \Delta_{j-1}(7) E_t^s r^n_{i_t} + \Delta_{j-1}(8) E_t^s u_t
\]

To solve for the first order condition on interest rate, first write equilibrium variables in terms of \(i_t\):

Beliefs:

\[
E_t^s r^n_{i_t} = (\phi(1 - K_{11}F_1 - K_{12}) - K_{11}F_2) E_t^{s-1} r^n_{i_{t-1}} - (K_{11}F_4 + \phi^u (K_{11}F_3 + K_{13})) E_t^{s-1} u_{t-1} \tag{B.30}
\]

\[
+ K_{12} r^n_{i_t} + K_{13} u_t + K_{11} i_t
\]

\[
E_t^s u_t = (\phi^u (1 - K_{21}F_3 - K_{23}) - K_{21}F_4) E_t^{s-1} u_{t-1} - (\phi (K_{21}F_1 + K_{22}) + K_{21}F_2) E_t^{s-1} r^n_{i_{t-1}} \tag{B.31}
\]

\[
+ K_{22} r^n_{i_t} + K_{23} u_t + K_{21} i_t
\]

Output gap:

\[
\hat{y}_t = \Xi(1) E_t^s r^n_{i_t} + \Xi(2) E_t^s u_t - \frac{1}{\sigma} \hat{y}_t + \frac{1}{\sigma} \frac{1}{1 - \phi} r^n_{i_t} \tag{B.33}
\]

Inflation:

\[
\pi_t = \kappa \theta \hat{y}_t + \Xi(3) E_t^s r^n_{i_t} + \Xi(4) E_t^s u_t - (1 - \theta) \frac{\kappa}{\sigma} \hat{y}_t + u_t \tag{B.34}
\]

Substitute the above endogenous variables into the first order condition on \(i_t\):

\[
\lambda_1 E_t^s r^n_{i_t} + \lambda_2 E_t^s u_t + \lambda_3 r^n_{i_t} + \lambda_4 u_t + \lambda_5 i_t = 0 \tag{B.35}
\]
where

\[
\lambda_1 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \Xi(1) + \frac{\partial \pi_t}{\partial i_t} \Xi(3) + \frac{1}{2} \Sigma \beta^j (\Delta_{j-1}(4) + \Delta(7)) \tag{B.36}
\]

\[
\lambda_2 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \Xi(2) + \frac{\partial \pi_t}{\partial i_t} \Xi(4) + \frac{1}{2} \Sigma \beta^j (\Delta_{j-1}(3) + \Delta(8)) \tag{B.37}
\]

\[
\lambda_3 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma} \frac{1}{1 - \phi} + \frac{1}{2} \Sigma \beta^j (\Delta_{j-1}(1) + \Delta(5)) \tag{B.38}
\]

\[
\lambda_4 = \frac{\partial \pi_t}{\partial i_t} + \frac{1}{2} \Sigma \beta^j (\Delta(2) + \Delta(6)) \tag{B.39}
\]

\[
\lambda_5 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \left( -\frac{1}{\sigma} \right) + \frac{\partial \pi_t}{\partial i_t} \left( -(1 - \theta) \frac{\kappa}{\theta} \right) \tag{B.40}
\]

and partial derivatives are derived as:

\[
\frac{\partial E^s t^n}{\partial i_t} = K_{11} \tag{B.41}
\]

\[
\frac{\partial E^s t^n u_t}{\partial i_t} = K_{21} \tag{B.42}
\]

\[
\frac{\partial \hat{y}_t}{\partial i_t} = \Xi(1) \frac{\partial E^s t^n}{\partial i_t} + \Xi(2) \frac{\partial E^s t^n u_t}{\partial i_t} - \frac{1}{\sigma} \tag{B.43}
\]

\[
\frac{\partial \pi_t}{\partial i_t} = \kappa \theta \frac{\partial \hat{y}_t}{\partial i_t} + \Xi(3) \frac{\partial E^s t^n}{\partial i_t} + \Xi(4) \frac{\partial E^s t^n u_t}{\partial i_t} - (1 - \theta) \frac{\kappa}{\sigma} \tag{B.44}
\]

Further substitute \( E^s t^n \) and \( E^s t^n u_t \):

\[
0 = \lambda_1 \left\{ (\phi (1 - K_{11}F_1 - K_{12}) - K_{11}F_2) E^s_{t-1} r^n_{t-1} - (K_{11}F_4 + \phi' (K_{11}F_3 + K_{13})) E^s_{t-1} u_{t-1} + K_{12} r^n_{t} + K_{13} u_{t} + K_{11} i_{t} \right\}
\]

\[
+ \lambda_2 \left\{ (\phi' (1 - K_{21}F_3 - K_{23}) - K_{21}F_4) E^s_{t-1} u_{t-1} - (\phi (K_{21}F_1 + K_{22}) + K_{21}F_2) E^s_{t-1} r^n_{t-1} + K_{22} r^n_{t} + K_{23} u_{t} + K_{21} i_{t} \right\}
\]

\[
+ \lambda_3 r^n_{t} + \lambda_4 u_{t} + \lambda_5 i_{t}
\]

The above equation solves the optimal nominal interest rate. Comparing with the guessed form yields the solution of \([F_1, F_2, F_3, F_4]\)
\[ F_1 = -\frac{\lambda_1 K_{12} + \lambda_2 K_{22} + \lambda_3}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \]  
(B.45)

\[ F_2 = -\frac{\lambda_1 (\phi (1 - K_{11} F_1 - K_{12}) - K_{12} F_2) - \lambda_2 (\phi (K_{21} F_1 + K_{22}) + K_{21} F_2)}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \]  
(B.46)

\[ F_3 = -\frac{\lambda_1 K_{13} + \lambda_2 K_{23} + \lambda_4}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \]  
(B.47)

\[ F_4 = -\frac{-\lambda_1 (K_{11} F_4 + \phi'' (K_{11} F_3 + K_{13})) + \lambda_2 (\phi'' (1 - K_{21} F_3 - K_{23}) - K_{21} F_4)}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \]  
(B.48)

I iterate the process until the conjectured interest rate function matches the above solution.
C Proofs

C.1 Proof of Lemma 2

The first (change of slope) and second point (intercept after cost push shock) are obvious. The following shows the proof for the their point (the sign of intercept after natural rate shock). The sign depends on the combination of parameter values and expectation on interest rate reaction function. Specifically, from Kalman Filtering, we know that \( K_{11} F_1 + K_{21} F_3 = 1 \), which indicates \( 0 < \frac{\partial E_{st}^r}{\partial r_n} < 1 \) and \( 0 < \frac{\partial E_{st}^u}{\partial u} < 1 \) as long as both \( F_1 \) and \( F_3 \) are non-zero. However, \( K_{11} = \frac{\partial E_{st}^r}{\partial i} \) may be greater or less than one, depending on whether \( F_1 \) is greater or smaller than 1. Consequently, the sign of the impact of \( r_t^* \) on the intercept of Phillips Curve under imperfect information, \( \left\{ (1 - \theta) \frac{\xi}{\sigma} (K_{11} - 1) + \frac{1 - \theta}{\theta} K_{21} \right\} \) is ambiguous.

C.2 Second Order Approximation of Household’s Utility Function

Follow Woodford (2003), Gali (2010), Walsh (2010) to prove that maximizing the utility of household is equivalent, up to second order approximation, to

\[
W = -\frac{1}{2} E_0 \Sigma \beta' \left( (e^{-1} + \phi) e^2 \text{var}_j(p_t(j)) + (\sigma + \phi) \bar{y}_t^2 \right) \tag{C.1}
\]

The next step is to prove the relationship between \( \text{var}_j(p_t(j)) \) with \( \text{var}(\pi_t) \). Denote \( \Delta_t = \text{var}_j[\log p_{jt}] \). Since \( \text{var}_j\bar{P}_{t-1} = 0 \), we have

\[
\Delta_t = \text{var}_j[\log p_{jt} - \bar{P}_{t-1}]
\]

\[
= E_j[\log p_{jt} - \bar{P}_{t-1}]^2 - [E_j\log p_{jt} - \bar{P}_{t-1}]^2
\]

\[
= E_j[\log p_{jt-1} - \bar{P}_{t-1}]^2 + (1 - \theta)(\int p_{tj}^* - \bar{P}_{t-1})^2 - (\bar{P}_t - \bar{P}_{t-1})^2
\]

As noted in Appendix A, \( \bar{P}_t = (1 - \theta) \int p_{tj}^* + \theta \bar{P}_{t-1} \), we have \( (1 - \theta) \int \log p_{tj}^* + \theta \bar{p}_{t-1}, \) which implies that \( (1 - \theta) \int \log p_{tj}^* - (1 - \theta) p_{tj} = \bar{p} - \bar{p}_{t-1} \). So, we have:

\[
\int \log p_{tj}^* = \left( \frac{1}{1 - \theta} \right) (\bar{p}_t - \bar{p}_{t-1}) \tag{C.3}
\]

Substitute this into (D.2) and get \( \Delta_t = \theta \omega_{t-1} + \left( \frac{\theta}{1 - \theta} \right) (\bar{p} - \bar{p}_{t-1})^2 \). Applying the definition of inflation results in:

\[
E_t \Sigma \beta' \Delta_t = \frac{\theta}{(1 - \theta)(1 - \theta \beta)} E_t \Sigma \beta' \pi_t^2 + t.i.p. \tag{C.4}
\]
C.3 Proof of Lemma 4

To prove the optimal response for a discretionary rate, express out the inflation expression in Lemma 3, \( \pi_t = -R\hat{y}_t \) as follows:

\[
\kappa \hat{y}_t + \Omega_{rr} (K_r - 1) r^n_t + \Omega_{ru} K_u + u_t = -R\hat{y}_t \quad (C.5)
\]

Then, substitute \( \hat{y}_t \) and \( \pi_t \) by interest rate, which results in:

\[
\left[ (R + \kappa) \frac{1}{\sigma} - \Omega_{rr} \right] r^n_t + u_t = \left[ (R + \kappa) \frac{1}{\sigma} - (\Omega_{rr} K_r + \Omega_{ru} K_u) \right] i_t \quad (C.6)
\]

To prove that \( F_r > 1 \) is equivalent to prove \( \Omega_{rr} K_r + \Omega_{ru} K_u > \Omega_{rr} \) which is just Assumption 1.

To prove that \( F_u < \theta F_u^p \) is equivalent to prove:

\[
(R^p + \kappa) \frac{1}{\sigma} < (R + \kappa) \frac{1}{\sigma} - (\Omega_{rr} K_r + \Omega_{ru} K_u)
\]

\[
\Leftrightarrow \Omega_{rr} K_r + \Omega_{ru} K_u < \frac{1}{\sigma} (R - R^p)
\]

As \((\hat{y}_t, \pi_t)\) is chosen optimally by the central bank, we know that \( R \cdot \text{slope} = \omega \) where \( \omega \) is in the CB’s objective function:

\[
\pi_t^2 + \omega \hat{y}_t^2
\]

So,

\[
\frac{1}{\sigma} \left( \frac{\omega}{R^p} - \frac{\omega}{R} \right) = \text{left}((1 - \theta) \frac{\kappa}{\sigma} K_r + \frac{1 - \theta}{\theta} K_u)
\]

(C.7)

Importantly, the RHS is just the difference between \( R_p \) and \( R \), as it measures the sensitivity inflation to beliefs.

Substitute this into the above equation leads to:

\[
\frac{\omega}{R^p} - \frac{\omega}{R} < R - R^p
\]

\[
\Leftrightarrow \omega < R^p \cdot R
\]

As Lemma 3 suggests that \( R > R^p \), the sufficient condition for the above inequality to hold is that \( R > \bar{R} \equiv \sqrt{\omega} \), where \( \omega \) is defined as the second order approximation of household utility.

C.4 Proof of Proposition 1

The proof of Proposition 1 take two steps. First I show that evaluated at the equilibrium optimizing discretionary interest rate, the partial derivative of combined Kalman gains weighted by their
effects on inflation is negative with respect to $F_r$ and is positive with respect to $F_u$.

**Step 1:** Under the assumption that $\sigma_r = \sigma_u$, I prove the following result.

Result 1. $\Omega_r \frac{\partial K_r}{\partial F_r} + \Theta_u \frac{\partial K_u}{\partial F_u} < 0$

Result 2. $\Omega_r \frac{\partial K_r}{\partial F_r} + \Theta_u \frac{\partial K_u}{\partial F_u} > 0$

The partial derivatives of Kalman gains on $F$:

\[
\frac{\partial K_r}{\partial F_r} = D^{-1} \left( F_r^2 \sigma_r^2 \sigma_u^2 - F_r^2 \sigma_r^2 \sigma_u^2 \right)
\]
\[
\frac{\partial K_u}{\partial F_r} = D^{-1} \left( F_r^2 \sigma_r^2 \sigma_u^2 - F_u^2 \sigma_r^2 \sigma_u^2 \right)
\]
\[
\frac{\partial K_u}{\partial F_u} = D^{-1} \left( 2F_r F_u \sigma_r^2 \sigma_u^2 \right)
\]
\[
\frac{\partial K_r}{\partial F_u} = D^{-1} \left( 2F_r F_u \sigma_r^2 \sigma_u^2 \right)
\]

where $D = F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2$.

To show Result 1, notice that $\frac{\partial K_r}{\partial F_r} < 0$, so it becomes sufficient to show $\frac{\partial K_u}{\partial F_r}$. This inequality holds when $F_u < F_r$ under discretion. From Lemma 4, we know that $F_r > F_r^p = 1$, and $F_u < \theta F_u^p$. In addition, we can derive the optimizing discretionary interest rate under perfect information to be: $F_u^p = \frac{x^2}{1 + \kappa^2} \sigma$, which leads to $F_u < 1$. Under $\sigma_r = \sigma_u$ and $F_r > F_u$, we have $\frac{\partial K_r}{\partial F_r} < 0$.

To show Result 2 is involves more steps. First, we know that $\frac{\partial K_r}{\partial F_u} = 0$, and under parameter specifications, $0 < \Theta_r < \Theta_u$. Thus, the sufficient condition for Result 2 to be hold is $\Theta_u \frac{\partial K_r}{\partial F_u} + \Theta_u \frac{\partial K_u}{\partial F_u} > 0$, which is equivalent to prove that $\frac{\partial K_r}{\partial F_u} + \frac{\partial K_u}{\partial F_u} > 0$.

From the partial derivatives of $K$ to $F$, and with the assumption $\sigma_r = \sigma_u$, we have

\[
\frac{\partial K_r}{\partial F_u} + \frac{\partial K_u}{\partial F_u} = D^{-1} \left( F_r^2 \sigma_r^2 \sigma_u^2 - F_u^2 \sigma_r^2 \sigma_u^2 - 2F_r F_u \sigma_r^2 \sigma_u^2 \right)
\]

Holding $F_r$ fixed, the quadratic function $f(F_u) = F_r^2 \sigma_r^2 \sigma_u^2 - F_u^2 \sigma_r^2 \sigma_u^2 - 2F_r F_u \sigma_r^2 \sigma_u^2$ reaches maximum at negative value of $F_u$. The range of $F_u$ is $[0, F_r]$ (Lemma 4). So, I only need to show $f'(0) > 0$ and $f(F_r) < 0$.

\[
f(0) = F_r^2 \sigma_r^2 \sigma_u^2 > 0
\]
\[
f(F_r) = -2F_r F_u \sigma_r^2 \sigma_u^2 < 0
\]

This completes the prove of Result 2.

**Step 2:** I prove that Result 1 and Result 2 makes Proposition 2 to hold.

**Part 1,** $F_r^c > F_r^d$

Suppose on the contrary that $F_r^c = F_r^d$. Then, express out the first order condition on $F_r$ after
\( r^n \) shock for both discretionary central bank and central bank with commitment, which is \( \pi_t \frac{\partial \pi_t}{\partial F_r} + \omega \frac{\partial \hat{y}_t}{\partial F_r} = 0 \). As \( \Omega_u > 0 \) and \( D > 0 \)

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_r \frac{\partial K_r}{\partial F_r} F_r + \Omega_u K_u + \Omega_u \frac{\partial K_u}{\partial F_r} F_r |_{\text{comm}} = \frac{\omega \hat{y}_t}{\sigma \pi_t} |_{\text{comm}} \\
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u |_{\text{disc}} = \frac{\omega \hat{y}_t}{\sigma \pi_t} |_{\text{disc}}
\]

(C.11) (C.12)

Under Assumption 1, The LHS of E.1 is smaller than the LHS of E.2, which makes \( \hat{y}_t \) to be smaller for commitment central bank. Substitute \( \hat{y}_t = -\frac{1}{\sigma} (F_r - 1) \), we have

\[
\frac{F_c^r}{\pi_t^c} > \frac{F_u^d}{\pi_t^d}
\]

(C.13)

The necessary condition for this inequality to hold is 1: \( F_c^r > F_u^d \), or \( \pi_t^c < \pi_t^d \).

Next, write out the expression for \( \pi_t^c < \pi_t^d \):

\[
-\frac{1}{\sigma} (F_r - 1) + \Omega_r K_r F_r + \Omega_u K_u F_r |_{\text{comm}} < -\frac{1}{\sigma} (F_r - 1) + \Omega_r K_r F_r + \Omega_u K_u F_r |_{\text{disc}}
\]

(C.14)

The necessary condition for this inequality to hold is: either \( F_c^r > F_u^d \), or \( \Omega_r K_r F_r + \Omega_u K_u F_r |_{\text{comm}} < \Omega_r K_r F_r + \Omega_u K_u F_r |_{\text{disc}} \). Under Assumption, this implies \( F_c^r > F_u^d \). In conclusion, we have \( F_r^c > F_r^d \).

**Part 2, \( F_u^c < F_u^d \)**

Suppose on the contrary that \( F_u^c = F_u^d \). Then, express out the first order condition on \( F_u \) after \( u_t \) shock for both discretionary central bank and central bank with commitment, which is \( \pi_t \frac{\partial \pi_t}{\partial F_u} + \omega \frac{\partial \hat{y}_t}{\partial F_u} = 0 \)

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_r \frac{\partial K_r}{\partial F_u} F_u + \Omega_u K_u + \Omega_u \frac{\partial K_u}{\partial F_u} F_u |_{\text{comm}} = \frac{\omega \hat{y}_t}{\sigma \pi_t} |_{\text{comm}} \\
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u |_{\text{disc}} = \frac{\omega \hat{y}_t}{\sigma \pi_t} |_{\text{disc}}
\]

(C.15) (C.16)

Under Assumption 2, The LHS of E.5 is smaller than the LHS of E.6, which makes \( \frac{\hat{y}_t}{\pi_t} \) to be bigger for commitment central bank. Substitute \( \hat{y}_t = -\frac{1}{\sigma} F_u u_t \), we have

\[
\frac{F_u^c}{\pi_t^c} < \frac{F_u^d}{\pi_t^d}
\]

(C.17)

The necessary condition for this inequality to hold is 1: \( F_u^c < F_u^d \), or \( \pi_t^c > \pi_t^d \).
Next, write out the expression for $\pi_t^c < \pi_t^d$ after $u_t$ shock:

$$-\frac{1}{\sigma}F_u + \Omega_r K_r F_u + \Omega_u K_u F_u|_{\text{comm}} < -\frac{1}{\sigma}F_u + \Omega_r K_r F_u + \Omega_u K_u F_u|_{\text{disc}} \quad (C.18)$$

The necessary condition for this inequality to hold is: either $F_u^c < F_u^d$, or $\Omega_r K_r F_u + \Omega_u K_u F_u|_{\text{comm}} < \Omega_r K_r F_u + \Omega_u K_u F_u|_{\text{disc}}$. Under Assumption 2, this implies $F_u^c < F_u^d$. In conclusion, we have $F_u^c > F_u^d$.

### C.5 Proof of Proposition 2

As analyzed in Section 3.3.2, the optimal polity rule satisfies:

$$-\frac{\kappa}{\sigma} + \Omega_r \left( K_r + \frac{\partial K_r}{\partial F_r} F_r \right) + \Omega_u \left( K_u + \frac{\partial K_u}{\partial F_u} F_u \right) = \frac{\omega}{\sigma} \tilde{y}_t \quad (C.19)$$

$$-\frac{\kappa}{\sigma} + \Omega_r \left( K_r + \frac{\partial K_r}{\partial F_r} F_r \right) + \Omega_u \left( K_u + \frac{\partial K_u}{\partial F_u} F_u \right) = \frac{\omega}{\sigma} \tilde{y}_t \quad (C.20)$$

After $K_r$ and $K_u$ are determined, the optimal one-time discretionary interest rate aims to minimize the ex-post loss:

$$\pi_t^2 + \omega \tilde{y}_t^2 \quad (C.21)$$

subjected to the constraint that $\pi_t = \{ \kappa - \sigma (\Omega_r K_r + \Omega_u K_u) \} \tilde{y}_t + (\Omega_r (K_r - 1) + \Omega_u K_u) r_t + u$. The first order condition of interest rate satisfies:

$$-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u = \frac{\omega}{\sigma} \frac{\tilde{y}_t}{\pi_t} \quad (C.22)$$

As long as $F_r \neq 0$ and $F_u \neq 0$, equation (C.16) is different from equation (C.13) and (C.14), which implies that one-time discretionary interest rate is different from the response of interest rate under optimal policy rule. Since the one-time discretionary interest rate is derived from the ex-post loss minimization problem, it implies that there is always a profitable deviation.