Lecture Note 13: Power Flows and Equilibrium Pricing

Christian Kroer*

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1 Introduction

This lecture note introduces a new topic: electricity markets, and their associated optimization problems. As we shall see, both economics and optimization play a key role in modern electricity grids.

For the first hundred years or so of the existence of the US power grid, it was managed by what are called *vertically integrated* utilities. These were companies that generated, sold, and transferred electricity directly to users. Typically these would also be monopolies, meaning that they were the only possible supplier in a given region. In contrast, the late 1990's and early 2000's saw what's usually referred to as the *deregulation*[1] of the electric grid,

In the deregulated markets, the choice of who generates what is made using auction-based mechanisms where the auctioneer is an *independent system operator* (ISO). ISOs are quasi-governmental entities whose charter is to operate the grid, including deciding who generates what using auctions. The overarching setup is very complicated, because e.g. the New York market uses two electricity auctions: a spot auction every five minutes (which decides on the allocation of generation and purchasing for the next five minutes), and a day-ahead auction every hour (which allocates power generation and purchasing for that hourly interval of the following day), as well as several *capacity auctions* meant to ensure that the grid has sufficient generation capacity. We will focus more on these auctions in the next lecture notes. First, this note will introduce the operational optimization problems that ISOs need to solve on a continuous basis.

Compared to a normal markets, the electric grid has many peculiarities. For example:

1. The grid operates in a continuous fashion, whereas the spot markets are operating every 5 minutes.

2. Supply (power generation) and demand (load generated by users) must be balanced at all times. The system will collapse if these quantities are not kept in check.

3. Goods (electricity) is generated at particular locations, and must be "transported" to the point of usage, potentially with a loss in power, or congestion of the wires

4. Electricity should be thought of as a "flow" in a network; therefore it's generally not possible to say that a particular user takes electricity from a particular plant. Both simply take electricity in and out of the "pool."

*Department of Industrial Engineering and Operations Research, Columbia University. Email: christian.kroer@columbia.edu.

[1]This name is arguably misleading, as the electric grid is actually a highly regulated industry still. A better term would perhaps be restructuring or decompositioning.

5. Different types of electricity generators (e.g. wind, gas, nuclear) all have very different operating constraints, and thus differ in their ability to increase or decrease productions, and the speed at which they can do so.

These peculiarities are good to keep in mind when thinking about the grid and its markets, because they mean that e.g. incentives can be a tricky subject.

2 Optimal Power Flow

We now introduce the *optimal power flow* (OPF) problem. In OPF, we are given a directed network V, E of nodes and edges representing the electric grid in question. The set of nodes V in power parlance is called the set of *buses*. I will use nodes and buses interchangeably. The buses should be thought of as important locations in the physical grid, e.g. generation points, load points, or substations. The set of edges E is the connections between buses. In power parlance, these are called transmission *lines*. We let E_i be the set of edges departing bus i.

The alternating current OPF (ACOPF) problem is a nonconvex quadratic optimization problem which models physics of the power flow problem including the fact that complex variables are needed. In particular, the net addition or removal of flow at a bus i will be a complex variable $p_i + \mathbf{i}q_i$, and similarly the power flow on a line $(i, j) \in E$ will be a complex variable $p_{ij} + \mathbf{i}q_{ij}$. We will mostly work with a linearization of this model, but I want to briefly describe it, so that you are aware of the approximation that is being made in the eventual LP we will use. To represent the problem, we will need the following variables:

- v_i is a complex number describing the voltage at bus $i \in V$
- p_i is a real number describing the difference between generation and demand of *real* power at bus $i \in V$
- q_i is the complex part of the difference between generation and demand of *reactive* power at bus $i \in V$
- p_{ij} is the real part of the power flow on line $i, j \in E$; $p_{ij} > 0$ means power is flowing from i to j and $p_{ij} < 0$ means power flows the opposite direction
- q_{ij} is the reactive power flow on line $i, j \in E$

We will also need the following constants:

- $\mathbf{i}$ will refer to the imaginary unit satisfying $\mathbf{i}^2 = 1$
- $y_{ij} = g_{ij} + \mathbf{i}b_{ij}$ is a complex number describing the *admittance* of the line i to j
- $\underline{v}_i, \overline{v}_i$ are lower and upper bounds on the voltage at bus i
- Each bus $i \in V$ is subject to box constraints on its real power $\underline{p}_i, \overline{p}_i$, and reactive power $\underline{q}_i, \overline{q}_i$
- Each line $i, j \in E$ is subject to a bound $\bar{s}_{ij}$ on the apparent power flow $p_{ij}^2 + q_{ij}^2$

With all that, the ACOPF problem looks as follows, where f is some objective functions that we wish to optimize subject to the power flow constraints.

$$
\begin{aligned}
\min_{v,p,q} \ & f(v,p,q) \\
\text{s.t. } & p_{ij} + \mathbf{i} q_{ij} = v_i(v_i^* - v_j^*) y_{ij}^*, && \forall (i,j) \in E \\
& p_{ij}^2 + q_{ij}^2 \leq \overline{s}_{ij}, && \forall (i,j) \in E \\
& \sum_{j \in E_i} p_{ij} = p_i, && \forall i \in V \\
& \sum_{j \in E_i} q_{ij} = q_i, && \forall i \in V \\
& p_i \in [\underline{p}_i, \overline{p}_i], && \forall i \in V \\
& q_i \in [\underline{q}_i, \overline{q}_i], && \forall i \in V \\
& |v_i| \in [\underline{v}_i, \overline{v}_i], && \forall i \in V
\end{aligned} \tag{ACOPF}
$$

The above problem is a very difficult optimization problem. In particular, even if f is a linear function, the first constraint is a nonconvex quadratic constraint, which makes the problem NP-hard in general. This leads to numerous problems, including the fact that this problem is typically too hard to solve to optimality for real-world OPF problems. A second issue is the lack of strong duality, which is something that we will need later.

3 Linearized Power Flow

Going forward, we will work with a simplified model of power flows, which linearizes the nonconvex quadratic constraint in Eq. (ACOPF). We will call this model *DC power flow* (DCOPF), though this terminology is misleading, because it does not actually model direct-current power flows. Instead, it is simply a linearized approximation to AC power flows.

This model is obtained by making a number of simplifying assumptions of Eq. (ACOPF). First, because reactive power is negligible relative to real power, we set all reactive power variables to zero, meaning that we can remove all q variables and associated constraints.

Next, we write the complex variables using polar coordinates $v_i = |m_i| e^{\mathbf{i}\theta_i}$ for each i. Then, we get the following equation for the real part of the nonconvex equation:

$$
p_{ij} = g_{ij}|m_i|^2 - |m_i||m_j| \left(g_{ij} \cos(\theta_i - \theta_j) - b_{ij} \sin(\theta_i - \theta_j) \right).
$$

Then, we set all voltage magnitudes equal to one, i.e. $|m_i| = 1$. Finally, we set $g_{ij} = 0$ because $g_{ij} \ll b_{ij}$.

After making all these simplifications, the DCOPF problems has only linear constraints:

$$
\begin{aligned}
\min_{\theta,p} \ & f(\theta,p) \\
\text{s.t. } & p_{ij} = b_{ij}(\theta_i - \theta_j), && \forall (i,j) \in E \\
& \sum_{j \in E_i} p_{ij} = p_i, && \forall i \in V \\
& p_i \in [\underline{p}_i, \overline{p}_i], && \forall i \in V \\
& |p_{ij}| \leq \overline{s}_{ij}, && \forall (i,j) \in E
\end{aligned} \tag{DCOPF}
$$

If f is also a linear function, then Eq. (DCOPF) is an LP.

In the formulation given here, each node $i \in V$ has a single power flow p_i into it (if $p_i > 0$) or out of it (if $p_i < 0$).

4 Economic Dispatch

In practice, nodes are often thought of as locations that potentially have both generators and demands. While Eq. (DCOPF) is completely general, it will be more convenient to include these multiple types of generators and demands in the model. To that end, let Ψ_i^D be the set of demands at node i, where each demand $d \in \Psi_i^D$ has some utility u_d of receiving power, and some upper bound $\overline{p}_d$ on how much power they can consume. Similarly, let Ψ_i^g be the set of generators at node i, where each generator $g \in \Psi_i^G$ has some cost c_d of generating power, and a maximum generating capacity $\overline{p}_g$. If we now set our objective f to be equal to the social welfare of the resulting allocation, we get the following LP:

$$\begin{aligned}
\max_{\theta,p} \sum_{i\in V} & \left(\sum_{d\in\Psi_i^D} u_d p_d - \sum_{g\in\Psi_i^G} c_g p_g \right) & & \\
\text{s.t. } & p_{ij} = b_{ij}(\theta_i - \theta_j), & \forall (i,j) \in E & \quad (1)\\
& \sum_{j\in E_i} p_{ij} = \sum_{g\in\Psi_i^G} p_g - \sum_{d\in\Psi_i^D} p_d, & \forall i \in V & \quad (2)\\
& p_d \in [0, \overline{p}_d], & \forall i \in V, d \in \Psi_i^D & \quad (3)\\
& p_g \in [0, \overline{p}_g], & \forall i \in V, g \in \Psi_i^G & \quad (4)\\
& |p_{ij}| \le \overline{s}_{ij}, & \forall (i,j) \in E & \quad (5)
\end{aligned}$$

A solution of this LP is referred to as *economic dispatch* because it maximizes efficiency. It also has a market equilibrium interpretation: let λ_i^* be the dual variable associated to Eq. (2) in an optimal solution, i.e. an economic dispatch solution. Then λ_i^* can be thought of as the *locational marginal price* (LMP) of electricity at node i: each demand at i is charged this price, and each generator at i is paid this price per unit of electricity. In fact, a variant of this LP that takes into account additional operational constraints is used for pricing in many real-world electricity markets.

4.1 Market Equilibrium Properties for Generators and Demands

If we consider the Lagrangified problem using the λ_i^* dual variables, we get the problem

$$\begin{aligned}
\max_{\theta,p} \sum_{i\in V} & \left(\sum_{d\in\Psi_i^D} u_d p_d - \sum_{g\in\Psi_i^G} c_g p_g \right) + \sum_{i\in V} \lambda_i^* \left(\sum_{g\in\Psi_i^G} p_g - \sum_{d\in\Psi_i^D} p_d - \sum_{j\in E_i} p_{ij} \right) & & \\
\text{s.t. } & p_{ij} = b_{ij}(\theta_i - \theta_j), & \forall (i,j) \in E & \quad (6)\\
& p_d \in [0, \overline{p}_d], & \forall i \in V, d \in \Psi_i^D & \quad (7)\\
& p_g \in [0, \overline{p}_g], & \forall i \in V, g \in \Psi_i^G & \quad (8)\\
& |p_{ij}| \le \overline{s}_{ij}, & \forall (i,j) \in E & \quad (9)
\end{aligned}$$

Now, if we consider the problem faced by an individual generator $g \in \Psi_i^G$ for some node i, in order to maximize their own utility they would like to solve the problem

$$\begin{aligned} \max_{p_g} \ & (\lambda_i^* - c_g) p_g \\ \text{s.t.} \ & p_g \in [0, \overline{p}_g] \end{aligned} \tag{10}$$

But we can see that the Lagrangified LP decomposes along generators, in the sense that p_g appears only in its own constraint Eq. (8), and with the exact same coefficients as in the individual generator utility maximization problem. Thus, by stationarity conditions, we get that the value p_g^* from the economic dispatch solution is also optimal for the individual generator given λ_i^*. A completely analogous argument shows that each demand also maximizes its utility.

It follows from the above that the prices and allocation from economic dispatch constitute a market equilibrium.

4.2 Spatial Arbitrage

Finally, let us try to understand the transmission variables p_{ij} which also depend on λ_i in the objective of Eq. (8). Consider the following problem given the optimal λ^*:

$$\begin{aligned} \max_{p_{ij}} \ & \sum_{i \in V} \lambda_i^* \sum_{j \in E_i} p_{ij} \\ \text{s.t.} \ & p_{ij} = b_{ij}(\theta_i - \theta_j), \quad \forall (i,j) \in E \\ & |p_{ij}| \leq \overline{s}_{ij}, \qquad\quad \forall (i,j) \in E \end{aligned} \tag{11}$$

This can be thought of as a spatial arbitraging operation. Since $\sum_{j \in Ei} p_{ij} = \sum_{d \in \Psi_i^D} p_d - \sum_{d \in \Psi_i^G} p_g$, we know that $\lambda_i^* \sum_{j \in Ei} p_{ij}$ is the *excess* payment at node i, which can be either positive or negative. While individual line revenues for the arbitrageur may thus be positive or negative, we see that Eq. (11) maximizes all the possible ways of transferring power across the network, given the prices. By a similar argument as before, we see that the economic dispatch solution optimally solves the spatial arbitrage problem. Thus, if we let the transmission operator collect these excess payments, then the transmission operator acts as a spatial arbitrageur, who optimally tries to buy and sell power while satisfying the (linearized) transmission constraints.

4.3 Economic Dispatch as a Mechanism

The economic dispatch framework derived in this section gives us a way to use markets to allocate power consumption and generation:

- Have every demand and generator submit their utility per unit of electricity, along with the consumption and generation caps
- Compute an economic dispatch solution for who generates and consumes what
- Charge everyone according to the dual prices

This is how allocation and pricing is performed in many of the *spot markets* used by various ISOs. Spot markets run on a frequent basis (e.g. every five minutes), and determine generation and consumption for any *uncommitted* load and generation capacity. I stress the uncommitted part here, because some generators and demands will already have entered binding contracts on price and quantity in earlier markets, such as the day-ahead market.

We now investigate a few properties that would be nice to have for this market.

- **Truthfulness**: Unfortunately this mechanism is not truthful. To see this, note that while each participant acts optimally *given* the prices, they can themselves influence the prices. If one considers a network with a single node, then it is straightforward to see that some generator and demander end up being the two entities setting the marginal price. They could then misreport in order to shift this price.

- **Efficiency**: if the submitted bids are truthful, then we would get efficiency by definition of the economic dispatch model. That said, we already noted that this mechanism is easily seen to not be truthful. A second concern for efficiency is that we introduced a lot of approximations in order to arrive at an LP.

- **Budget balance**: The ISO needs to ensure that after paying generators and charging demands it ends up with a nonnegative amount of leftover money. However, we already saw in the spatial arbitrage section that the excess payments are captured via the p_{ij} variables, and the spatial arbitrager can make their utility at least zero, so revenue adequacy is guaranteed. ISOs are typically not allowed to make money either; for that reason the money made from spatial arbitrage is usually thought of as going to the providers of the transmission network, or towards additional investment in the network.

- **Individual rationality**: is every participant incentivized to participate in the market? This is easily seen to be true from the market equilibrium condition, as long as participants do not overstate their capacity, or report utilities/costs that are respectively higher/lower than their true values.

In addition to the approximations that we made going from ACOPF to DCOPF, this note also made some implicit assumptions. One of the biggest is that every generator can choose in continuous fashion how much electricity to produce. In practice, generators have various types of constraints on how they can change their output. For example, several types of energy producers require a long time to ramp up or down production (say up to a day), and they may have minimum generation levels for when they are turned on. This is the case for several traditional generators such as nuclear and coal. Natural gas also has similar constraints. This introduces a discrete nature into the problem: we may need a day or more to reach certain production levels, and so the real-time market is operating "too late" for some decisions to be made. This motivates the use of day-ahead markets, which we will study in the next lecture note.

Renewables also have different types of constraints on their production, that depend on the type of renewable. For example, wind generators are not necessarily able to adjust their output at all, and are thus required to produce electricity at whatever level the weather dictates. This can even lead to negative energy prices, depending on whether we have a cost-free way of handling excess power. All these constraints, as well as a general desire on the part of market participants for a certain amount of predictability in their revenues, necessitate additional market mechanisms that allow us to settle some generation and consumption further in advance than the spot market allows. This will be the topic of the next note.

5 Historical Notes

A good book on the optimization aspects of the power grid is Taylor [3]. This book also has some coverage of energy markets. Kirschen and Strbac [1] has extensive coverage of the economic aspects of energy system.

Sweeney [2] provides a detailed account of the California energy crisis, which is an interesting case study in how not to design an energy markets. That crisis lead to severe blackouts, huge budget deficits for several energy companies (with one going bankrupt), and had large ramifications for the state budget.

There are also several good courses available online in various formst. Penn State has lecture notes available[2] for an excellent introductory course on the power grid and markets. Jalal Kazempour from the Danish Technical University has a set of slides and lecture videos[3] that give a really nice optimization-based introduction.

References

[1] Daniel S Kirschen and Goran Strbac. *Fundamentals of power system economics.* John Wiley & Sons, 2018.

[2] James L Sweeney. *The California electricity crisis.* Hoover Press, 2013.

[3] Joshua Adam Taylor. *Convex optimization of power systems.* Cambridge University Press, 2015.

[2]Found here: `https://www.e-education.psu.edu/ebf483/`
[3]Found here: `https://www.jalalkazempour.com/teaching`