IEOR8100: Economics, AI, and Optimization
Lecture Note 11: Introduction to Auctions

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1 Introduction

In fair division we initially did not worry about the fact that we might not necessarily know the utility function u_i of each agent. We briefly studied settings where agents may misreport their valuation in the context of dominant-resource fairness. In this lecture note we continue the study of settings where we will worry about whether agents tell the truth or not. The general study of this type of setting is called *mechanism design*.

We will study the most classical mechanism-design setting: auctions. We will start by considering single-item auctions: there is a single good for sale, and there is a set of n buyers, with each buyer having some value v_i for the good. The goal will be to sell the item via a *sealed-bid* auction, which works as follows:

1. Each bidder i submits a bid $b_i \geq 0$, without seeing the bids of anyone else.
2. The seller decides who gets the good based on the submitted bids.
3. Each buyer i is charged a price p_i which is a function of the bid vector b.

A few things in our setup may seem strange. First, most people would not think of sealed bids when envisioning an auction. Instead, they typically envision what's called the *English auction.* In the English auction, bidders repeatedly call out increasing bids, until the bidding stops, at which point the highest bidder wins and pays their last bid. This auction can be conceptualized as having a price that starts at zero, and then rises continuously, with bidders dropping out as they become priced out. Once only one bidder is left, the increasing price stops and the item is sold to the last bidder at that price. This auction format turns out to be equivalent to the *second-price* sealed-bid auction which we will cover below. Another auction format, which is less prevalent in practice, is to start the price very high such that nobody is interested, and then continuously dropping the price until some bidder says they are interested, at which point they win the item at that price. The Dutch auction is likewise equivalent to the *first-price* sealed-bid auction, which we cover below.

Secondly, it would seem natural to always give the item to the highest bid in step 2, but this is not always done (though for now we will always use that rule). Thirdly, the pricing step allows us to potentially charge more bidders than only the winner. This is again done in some reasonable auction designs, though we will mostly focus on auction formats where $p_i = 0$ if i does not win.

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1.1 First-price auctions

First-price auctions are perhaps what most people imagine when we say that we are selling our good via a sealed-bid auctions. In first-price auctions, each buyer submits some bid $b_i \geq 0$, and then we allocate the item to the buyer i^* with the highest bid and charge that buyer b_{i^*}. This is also sometimes referred to as *pay-your-bid*.

Let's briefly try to reason about what might happen in a first-price auction. Clearly, no buyer should bid their true value for the good under this mechanism; in that case they receive no utility even when they win. Instead, buyers should *shade* their bids, so that they sometimes win while also receiving strictly positive utility. The problem is that buyers must strategize about how other buyers will bid, in order to figure out how much to shade by.

This issue of shading and guessing what other buyers will bid happened in early Internet ad auctions, where first-price auctions were initially adopted. *Overture* was an early pioneer in selling Internet sponsored search ads via auction. They initially ran first-price auctions, and provided services to MSN and Yahoo (which were popular search engines at the time). Bidding and pricing turned out to be very inefficient, because buyers were constantly changing their bids in order to best respond to each other. Plots of the price history show a clear "sawtooth pattern," where a pair of bidders will take turns increasing their bid by 1 cent each, in order to beat the other bidder. Finally, one of the bidders reaches their valuation, at which point they drop their bid much lower in order to win something else instead. Then, the winner realizes that they should bid much lower, in order to decrease the price they pay. At that point, the bidder that dropped out starts bidding 1 cent more again, and the pattern repeats. This leads to huge price fluctuations, and inefficient allocations, since about half the time the item goes to the bidder with the lower valuation.

All that said, it turns out that there does exist at least one interesting characterization of how bidding should work in a single-item first-price auction (the Overture example technically consists of many "independent" first-price auctions; though that independence does not truly hold as we shall see later).

For this characterization, we assume the following symmetric model: we have n buyers as before, and buyer i assigns value $v_i \in [0, \omega]$ for the good. Each v_i is sampled IID from an increasing distribution function F. F is assumed to have a continuous density f and full support. Each bidder knows their own value v_i, but only knows that the value of each other buyer is sampled according to F.

Given a bid b_i, buyer i earns utility $v_i - b_i$ if they win, and utility 0 otherwise. If there are multiple bids tied for highest then we assume that a winner is picked uniformly at random among the winning bids, and only the winning bidder pays.

It turns out that there exists a *symmetric equilibrium* in this setting, where each bidder bids according to the function

$$\beta(v_i) = \mathbb{E}[Y_1 | Y_1 < v_i],$$

where Y_1 is the random variable denoting the maximum over $n-1$ independently-drawn values from F.

Theorem 1. *If every bidder in a first-price auction bids according to β then the resulting strategy profile is a Bayes-Nash equilibrium.*

Proof. Let $G(y) = F(y)^{n-1}$ denote the distribution function for Y_1.

Suppose all bidders except i bids according to β. The function β is continuous and monotonically increasing: a higher value for v_i simply adds additional values to the highest end of the distribution. As a consequence, the highest bid other than that of bidder i is $\beta(Y_1)$. It follows that bidder i should never bid more than $\beta(\omega)$, since that is the highest possible other bid. Now consider bidding

$b_i \leq \beta(\omega)$. Letting z be such that $\beta(z) = b_i$, the expected value that bidder i obtains from bidding b_i is:

$$\begin{aligned}
\Pi(b_i, v_i) =& G(z)[v_i - \beta(z)] \\
=& G(z)v_i - G(z)\mathbb{E}[Y_1 | Y_1 < z] && \text{by definition of } \beta(z) \\
=& G(z)v_i - \int_0^z yg(y)dy && \text{by definition of expectation} \\
=& G(z)v_i - G(z)z + \int_0^z G(y)dy && \text{integration by parts} \\
=& G(z)(v_i - z) + \int_0^z G(y)dy
\end{aligned}$$

Now we can compare the values from bidding $\beta(v_i)$ and b_i:

$$\begin{aligned}
\Pi(\beta(v_i), v_i) - \Pi(b_i, v_i) &= G(v_i)(v_i - v_i) + \int_0^{v_i} G(y)dy - G(z)(v_i - z) - \int_0^z G(y)dy \\
&= G(z)(z - v_i) - \int_{v_i}^z G(y)dy
\end{aligned}$$

If $z \geq v_i$ then this is clearly positive since $G(z) \geq G(y)$ for all $y \in [v_i, z]$. If $z \leq v_i$, then $G(z) \leq G(y)$, and so we have a negative number and subtract a more negative number. □

A nice property that follows from the monotonicity of β is that the item is always allocated to the bidder with the highest valuation, and thus the symmetric equilibrium is efficient.

1.2 Second-price auctions

Now we look at another pricing rule: the *second-price auction*. The *second-price auction* turns out to simply allow buyers to submit their true value as their bid. In a second-price auction, the winning bidder i^* is charged the *second-highest bid*. It's easy to see that a bidder should simply bid their valuation in this auction format. There are four cases to consider for a non-truthful bid $b_i \neq v_i$:

1. $b_i > v_i \geq b_2$ where b_2 is the second-highest bid. In that case buyer i would have gotten the same utility from bidding v_i.
2. $b_i > b_2 > v_i$ where b_2 is the second-highest bid. In that case buyer i wins, but gets utility $v_i - b_2 < 0$, and they would have been better off bidding their valuation.
3. $b_i < b_2 < v_i$ where b_2 is the second-highest bid. In that case buyer i does not win, but they could have won and gotten strictly positive utility if they had bid their valuation.
4. $b_2 < b_i < v_i$ where b_2 is the second-highest bid. In that case buyer i wins, but they would have won, and paid the same, if they had bid their true value.

It follows that the second-price auction is strategyproof, which is also called *dominant-strategy incentive compatible*, because an agent should report their true valuation no matter what everybody else does.

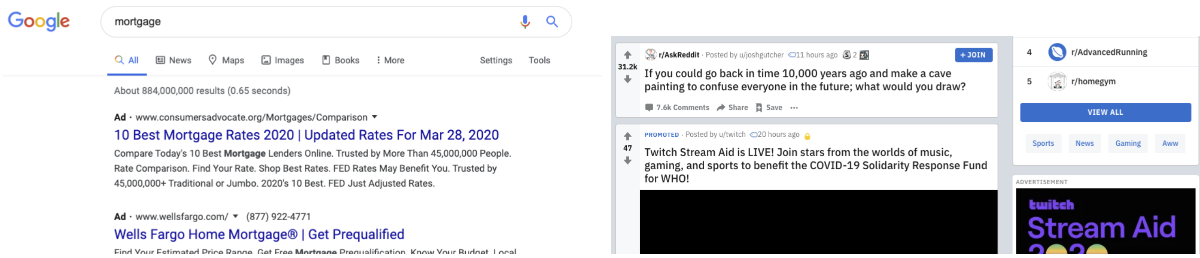

Figure 1: Left: A Google query for "mortgage" shows 2 ads. Organic search results follow further down. Right: The front page of Reddit. The second feed story is an ad.

1.3 Sponsored Search Auctions

First and second-price auctions are natural to think of due to traditional ideas of what auctions are. However, in the modern Internet era new types of auction settings have become prevalent that go beyond single-item auctions. This is largely due to Internet advertising, which funds essentially the entirety of Google as well as Facebook and other free major Internet services such as Twitter and Reddit. In these auctions there are multiple reasons why we cannot simply analyze single-item auctions as above. Two major reasons are: 1) advertisers participate in millions of auctions and have budgets that span across these auctions, and 2) each individual auction typically has multiple ad slots for sale. We will now investigate the second reason, while the first reason will be investigated in the next lecture note.

The classical example of a sponsored search auction is a Google query, where a few ads (typically 2) are shown at the top of the search. Figure 1 on the left shows an example search for the keyword "mortgage." The sponsored search auction model can also be used to approximate other settings such as the insertion of ads in a feed. For example, Reddit typically inserts 1 ad in the set of visible results before scrolling (see Figure 1 on the right), with another ad appearing in the next 10-15 results (tested March 28th 2020). Similarly, Facebook and Twitter insert 1-2 sponsored posts near the top of the feed. Truly capturing feed auctions does require some care, however. The assumption of there being a fixed number of items is incorrect for that setting. Instead, the number of ads shown depends on how far the user scrolls, the size of the ads, and what else is being shown in terms of organic content.

In the sponsored search auction model, a set of k *slots* are for sale. The slots are shown in ranked order, and the value that an advertiser derives from showing their ad in a particular slot j decomposes into two terms $v_{ij} = c_i q_j$ where c_i is the value that the advertiser places on a user clicking on their ad, and q_j is the advertiser-independent click probability of slot j. It is assumed that $q_1 \geq q_2 \geq \cdot \geq q_k$, i.e. the top slot is better than the second slot, and so on. It is assumed that advertisers are not inherently interested in getting their ad shown. Instead, their goal is to get the user to click on the ad. Hence, our auction design will only charge an advertiser if their ad is shown.

The *generalized second-price* (GSP) auction sells the k slots as follows: we collect a set of bids $b \in \mathbb{R}^n$ (assume $n \geq k$). Then we sort b (say in the order $b_1 \geq b_2 \geq \cdots \geq b_n$), and allocate the slots in order of bids (so b_1 gets slot 1, up to bid b_k getting slot k). If the user clicks on ad $i \leq k$, then advertiser i is charged the next-highest bid b_{i+1}. GSP generalizes second-price auctions in the sense that if $k = 1$ then this auction format is equivalent to the standard second-price auction (if we take expected values in lieu of the pay-per-click model).

2 Historical Notes

The issues with first-price in the context of Overture's sponsored search auctions is described in Edelman and Ostrovsky [1], which also shows plots from real data exhibiting the sawtooth pattern. The derivation of the symmetric equilibrium of the first-price auction follows the proof from Krishna [3]. Interestingly, first-price auctions have experiences a resurgence in the context of display advertising, where many independent ad exchanges moved to first price in 2018, and Google followed suit and moved their Ad Manager to first price in 2019[1].

The second-price auction is sometimes referred to as the *Vickrey auction* after its inventor [4]. The generalized second-price auction was described by Edelman et al. [2], though it had been in use in the Internet ad industry for a while at that point.

References

[1] Benjamin Edelman and Michael Ostrovsky. Strategic bidder behavior in sponsored search auctions. *Decision support systems*, 43(1):192–198, 2007.

[2] Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz. Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords. *American economic review*, 97(1):242–259, 2007.

[3] Vijay Krishna. *Auction theory.* Academic press, 2009.

[4] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16(1):8–37, 1961.

[1] see `https://www.blog.google/products/admanager/update-first-price-auctions-google-ad-manager/`