Worst Case Analysis

E.g. Heap

\[ n \text{ INSERTS } \quad \text{ign} \]
\[ n \text{ DELETE-MIN } \quad \text{ign} \]
\[ O(n \log n) \]

seq. of
\[ n \text{ INSERTS took } O(n) \]
Stack

Push
Pop

Multipop

$O(1)$ time

Push(1)
Push(2)
Push(3)
Pop()
Push(S)

Multipop(S, 2)

5 times

Push
Multipop(S6)
A Push, Pop, MP n ops each O(n²) time.

Can have 1 MP take $\Omega(n)$ time.

Can you have $\Omega(n)$ MPs each taking $\Omega(n)$ time.

Push MP MP MP Push MP MP MP Push...
Any sequence of $n$ Push Pop MP's take $O(n)$ time in total.

$O(n)$ time per op amortized

Amortized Analysis

3 methods of AA.
- Aggregate Analysis
- Banker's Method
- Potential Function Method.
Let $m(i) =$ # of pops done in $i$th multipop
Let $p =$ # of pushes done by the alg.

($\# \text{ pushes} \geq \# \text{ pops}$)

$\sum_{i} m(i) \leq p$

($\# \text{ pops}$)

$\text{Time} = \# \text{ pushes} + \# \text{ pops} + \sum_{i} m(i) \leq p + p + p \leq 3p \leq 3n$. 
Banke's

\[
\begin{array}{c|c|c}
\text{OPS} & \text{Rest. cost} & \text{Amort. cost} \\
\hline
\text{push} & 1 & \frac{\hat{C}_c}{2} \\
\text{pop} & 1 & 0 \\
\text{mp} & k & 0 \\
\end{array}
\]

Assign $\hat{C}_c$

Show $\sum_{i=1}^{n} \hat{C}_c \geq \sum_{i=1}^{n} C_i$ for any seq. of bpt.

Bound on n ops is $\sum_{i=1}^{n} \hat{C}_c \geq \sum_{i=1}^{n} C_i \leq 2n$
Proof that $\sum c_i \geq \sum c_i$

Push 0
Pop 0

Whenever you push() use $\$1$ to pay for the push, store the other $\$1$ in the "bank" to pay for the item being popped.

$\Rightarrow$ Balance always $\geq 0$. 
Potential

\[ \Phi(D_i) = \text{potential after } i\text{th op.} \]

Real costs \( C_i \)

Am costs \( \hat{C}_i = C_i + \Phi(D_i) - \Phi(D_{i-1}) \)

\[ \sum_{i=1}^{\hat{C}_i} = \sum_{i=1}^{C_i} + \sum_{i=1}^{\Phi(D_i) - \Phi(D_{i-1})} \]

\[ \sum_{i=1}^{\hat{C}_i} \geq \sum_{i=1}^{C_i} \Rightarrow \Phi(D_n) - \Phi(D_0) \geq 0 \]

\[ \Phi(D_n) \geq \Phi(D_0) \]

If \( \Phi(D_n) \geq \Phi(D_0) \) then \( \sum_{i=1}^{\hat{C}_i} \) is an upper bound on \( \sum_{i=1}^{C_i} \).
\[ \phi(D_i) = \# \text{items in stack} \]

\[ \phi(D_o) = 0 \]

\[ \phi(D_i) \geq 0 \quad \forall i \quad \phi \text{ is valid} \]

Compute \[ \hat{C}_i = C_i + \Delta \phi \]

\begin{align*}
\text{Push} & \quad \hat{C}_i = 1 + 1 = 2 \\
\text{Pop} & \quad \hat{C}_i = 1 + (-1) = 0 \\
MP(k) & \quad \hat{C}_i = k + (-k) = 0
\end{align*}

Cost of n ops \leq \sum_{i=1}^{n} \hat{C}_i \leq 2n
3 bit counter

\[
\begin{array}{cccc}
000 & \rightarrow & 0 & \rightarrow 20 \\
001 & & 1 & 2 \\
010 & & 2 & 2 \\
011 & & 1 & 2 \\
100 & & 3 & 2 \\
101 & & 2 & 2 \\
110 & & 2 & 2 \\
111 & & 1 & 2 \\
\end{array}
\]

charge $5$ per inc.
use $\$1$ to pay for new store $\$1$ to pay for flip back to $0$.

\[\text{total } \leq 2^n\]

\[\text{cost (increment)} = \# \text{ bits flipped}\]

\[k \text{ bit counter, each inc. costs } \leq k\]
Let $n$ be a power of 2

$n \equiv nC$

Least Significant Bit flipped $n$ times

next bit $\equiv n/2 \text{ time}$

$\quad \equiv n/4 \text{ time}$

Most Significant Bit flipped $n/2^{k-1}$ times

Total flips:

\[ n + \frac{n}{2} + \frac{n}{4} + \ldots \leq 2^n \]

$n \equiv nC \leq 2n - \text{total time}$

$O(1) \text{ op amortized}$
\( \phi(D_i) = \# 1's \text{ in } D_i \) center

\( \phi(D_0) = 0 \)

\( \phi(D_i) \geq 0 \quad \forall i \)

\( \widehat{C}_c = (C_i) + (\phi(D_i) - \phi(D_{i-1})) \)

let \( f_{0i} = \# \text{bits flipped } \quad 0 \geq i \)

\( f_{1i} = \text{... } \quad 1 \geq i \)

\( = (f_{0i} + f_{1i}) + (f_{0i} - f_{1i}) \)

\( = 2f_{0i} \quad \text{Total cost for} \)

\( \leq 2 \quad \text{ops is } 27. \)