SAT is NP-complete.

3-SAT: SAT but exactly 3 literals / clause (literal = var. or negation of variable)

\[ \phi = (x_1 \lor \overline{x}_2 \lor \overline{x}_3) \land (x_4 \lor x_1 \lor x_2) \land (x_2 \lor x_5 \lor \overline{x}_3) \land (x_1 \lor x_5 \lor \overline{x}_6) \]

\( n \) vars.
\( m \) clauses

3SAT is a special case of SAT
1-SAT  \( x_1 \land x_2 \land x_3 \land x_4 \ldots \)

is easy

2-SAT  \( (x_1 \lor x_2) \land (x_1 \lor \neg x_2) \land (x_3 \lor x_4) \ldots \)

is poly time.

Show 3-SAT is NP-complete

1. 3-SAT \( \in \text{NP} \)
2. Choose SAT to reduce from
3. Given an instance of SAT, show how to create an instance of 3-SAT
Describe \( f \), walks clauses by clause

\( k \equiv \# \) literals in a clause

if \( k = 1 \)

\[
\begin{align*}
X \xrightarrow{f} & (x_1, v x_1, v x_1) \\
\end{align*}
\]

if \( k = 2 \)

\[
\begin{align*}
(x_1, v x_2) \xrightarrow{f} & (x_1, v x_2, v x_2) \\
\end{align*}
\]

if \( k = 3 \)

\[
\begin{align*}
(x_1, v x_2, v x_3) \xrightarrow{f} & (x_1, v x_2, v x_3) \\
\end{align*}
\]

if \( k = 4 \)
\[ C = (x_1 \lor x_2 \lor x_3 \lor y_1) \rightarrow C' \equiv (x_1 \lor x_2 \lor y_1) \land (\overline{y_1} \lor \overline{y_1} \lor x_3 \lor x_4) \]

For any setting of the \( x_i \)'s, \( C \) is true iff \( C' \) can be true.

\[ \Rightarrow \text{ if } C \text{ is true, at least one literal } x_1, x_2, x_3, x_4 \text{ is true. If } x_1 \text{ or } x_2 \text{ is true set } y_1 = \overline{F} \text{ then } \overline{y_1} \text{ is true & both clauses of } C' \text{ are true.} \]

\[ \text{If } x_3 \text{ or } x_4 \text{ is true, set } y_1 = T, \text{ both clauses of } C' \text{ are true.} \]

\[ \Leftarrow \text{ if } C \text{ is false, } x_1 = x_2 = x_3 = x_4 = F. \text{ Then } C' \text{ reduces to } y_1 \land \overline{y_1}, \text{ which must be false.} \]
\[ k = 5 \]
\[ (x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5) \rightarrow\]
\[ \begin{align*}
  & F \\
  & F \\
  \lor (x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5) \\
  \land (\neg g_1 \lor \neg x_3 \lor x_4) \\
  \land (\neg g_2 \lor \neg y_1 \lor x_5) \\
  \end{align*} \]
\[ (x_1 \lor x_2 \ldots \lor x_k) \rightarrow (x_1 \lor x_2 \lor y_1) \]
\[ \land (\neg g_1 \lor \neg x_3 \lor y_2) \]
\[ \land (\neg g_2 \lor \neg y_1 \lor y_3) \]
\[ \ldots \]
\[ \land (\neg g_{k-3} \lor x_{k-1} \lor x_k) \]
Each clause in \( C \) \( \rightarrow \) at most \( n \) clauses in \( C' \)

\( k \leq n \)

SAT
\( n \) vars.
\( m \) clauses

3 SAT
\( \rightarrow \) vars \( \leq n + m n \)
clauses \( \leq m n \)

\( f \) is poly time
2SAT ≤ 3SAT

To show 2-SAT is NP-comp.

Show SAT ≤ 2-SAT

\[(x_1 \lor x_2 \lor y_3) \quad (x_1, y_1)\]

\[(\bar{y}_1, x_2)\]
SAT

\{\text{poly time}\}

3SAT ( )

\{\text{poly time}\}
∀X ∈ NP \ X ≤ₚ SAT
SAT ≤ₚ 3SAT

∴ ∀X ∈ NP \ X ≤ₚ 3-SAT
Clique Given a graph $G = (V, E)$ and an int. $k$, does $G$ have a $k$-clique, that is, a subset $V' \subseteq V$ with $|V'| = k$ s.t. $\forall x, y \in V' \ x \neq y$, edge $(x, y)$ exists.
Clique is NP-hard.

1) Clique $\in$ NP.

2) 3-SAT
\((x_1 \lor \bar{x_2} \lor x_3) \land (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_2 \lor x_3)\)

node for each appearance of var

edge between pairs of nodes that can simultaneously be true

\(k = \#\text{clauses}\)
$\emptyset$ is sat. iff $G$ has a $k$-clique

$\implies \emptyset$ is sat., given a sat. assignment, pick one literal per clause, by construction all the edges are there

$\iff$ Suppose $G$ has a $k$-clique, it must contain exactly 1 literal per clause, and by construction, all can be set to true simultaneously.