- Everyone take a $1
- for i = 1 to 1000
  follow some rule I make up

① What is the maximum amt. of money I can end up with?

~~A. (80 people)(1000 rounds) = $80,000~~ (weak upper bound)

B. $80 total (tighter upper bound)

Sometimes a standard analysis is too weak. Doesn't take into account restrictions on what can happen over a series of steps.

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Heap Sort

insert $O(\lg n)$ time.

n inserts consecutively

$O(n \lg n) \rightarrow O(n)$

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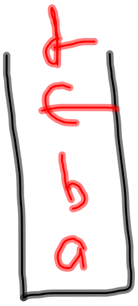
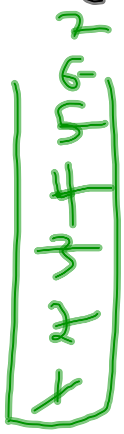
Stack

PUSH

POP



O(1) time



Push 1
Push 2
Push 3
Pop
Push 4
Multipop 2

Push 5
Pop
Push 6
Push 7
Multipop 5

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Push $O(1)$
Pop $O(1)$
Multipop $O(k)$

n operations

each operation takes $O(n)$ time
n ops
total time $= O(n^2)$

Example of n ops taking $\Omega(n^2)$ time.
push - need Multipops on large stacks

$\underbrace{P\,P\,P\,P\,P\,P\,P\,P\,P}_{k}$ MP(k) $\underbrace{P\,P\,P\,P\,P\,P\,P}_{k}$ MP(k)

$k = \frac{n}{2}$

Can't construct such an example.

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Any seq. of n Push, Pop, MP takes O(n) time.

Amortized time / op = $O(n)/n = O(1)$.

Amortized Analysis

3 methods

- Agregate Analysis
- Banker's Method
- Potential function analysis

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1) Agregate Analysis (Push, MP (MP(1) = Pop))

let $m(i)$ = # of pops in the i^{th} multipop

let p = # of pushes done overall

$\underline{P}\,\underline{PP}$ MP $\underline{PPP}$ MP $\underline{PPPP}$ MP

$$\sum_i m(i) \le p.$$

Total time = # pushes + time for all multipops

$$= p + \sum_i m(i)$$

$$\le p + p = 2p \le 2n \quad (p \le n)$$

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Banker's Method.

	Real Cost c_i	Am. Cost $\hat{c}_i$
Push	1	2
Pop	1	0
MP(k)	k	0

Conditions

$$\forall \ell, \sum_{i=1}^{\ell} \hat{c}_i \geq \sum_{i=1}^{\ell} c_i$$

Show $\sum_{i=1}^{n} \hat{c}_i \leq X$

$$\Rightarrow \sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \leq X$$

Prove $\sum_{i=1}^{\ell} \hat{c}_i \geq \sum_{i=1}^{\ell} c_i$

Proof Whenever you Push leave \$1 on the item, & that pays for the future Pop.
Because each item is Pushed before popped, I always have the \$ to pay $\Rightarrow \sum \hat{c}_i \geq \sum c_i$

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P1
P2
P3
Pop
P4
P5
MP(3)

~~5\$~~
~~4\$~~
~~3\$~~
~~2\$~~
1 \$

payed 6
\- Real cost 3
bank ~~3~~ 2

Total Am. Cost $\sum \hat{c}_i \leq 2n$

$$\therefore \sum c_i \leq 2n.$$

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Potential function

D_i = state of system after i^{th} operation

Define $\Phi(D_i)$ potential assoc. w/ D_i

Real costs c_i

Am. costs $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \cancel{\Phi(D_1)} - \Phi(D_0) + \cancel{\Phi(D_2)} - \cancel{\Phi(D_1)} + \cancel{\Phi(D_3)} - \cancel{\Phi(D_2)} \; \vdots \; + \Phi(D_n) - \cancel{\Phi(D_{n-1})}$$

$$\Rightarrow \sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

want $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i \Rightarrow \Phi(D_n) \geq \Phi(D_0)$

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If $\Phi(D_n) \geq \Phi(D_0)$ then $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$ &

$\sum_{i=1}^{n} \hat{c}_i$ is an upper bound.

(Typically we choose Φ s.t. $\Phi(D_0) = 0$
$\Phi(D_n) \geq 0$.)

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Choose $\Phi(D_i) = $ # items in stack.

$$\Phi(D_0) = 0$$
$$\Phi(D_n) \geq 0 \Rightarrow \Phi \text{ is a valid pot. func.}$$

Am. cost.

Push $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
$= 1 + 1 = 2$

Pop $\hat{c} = c_i + \Phi(D_i) - \Phi(D_{i-1})$
$= 1 + (-1) = 0$

MP(k) $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
$= k + (-k) = 0$

For any seq. of n Push, Pop, MP $\sum_{i=1}^{n} \hat{c}_i \leq 2n \rightarrow \sum_{i=1}^{n} c_i \leq 2n$ $\blacksquare$

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3 bit counter

0→1	1→0				cost
		0	0	0	
1	0	0	0	1	1
1	1	0	1	0	2
1	0	0	1	1	1
1	2	1	0	0	3
1	0	1	0	1	1
1	1	1	1	0	2
1	0	1	1	1	1
0	3				3

k-bit counter

cost (increment) = # bits flipped

n increments

flip $\leq nk$ bits overall.

Fact n increments, k bit counter, # flips $= O(n)$.

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Total bits flipped

$\cdots \frac{n}{8} \quad \frac{n}{4} \quad \frac{n}{2} \quad n$

total flips $\leq$

$$n + \frac{n}{2} + \frac{n}{4} + \cdots + 1$$

$$\leq n\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \right)$$

$$\leq 2n.$$

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Pot. func.

$\Phi(D_i) = \#$ 1's in the counter

$\Phi(D_0) = 0$

$\Phi(D_i) \geq 0 \Rightarrow \Phi$ valid

Def: $f_{01} =$ # bits flipped $0 \to 1$

$f_{10} =$ # bits flipped $1 \to 0$

$$\hat{c}_i = c_i + \overbrace{\Phi(D_i) - \Phi(D_{i-1})}^{\Delta}$$

$$= (f_{01} + \cancel{f_{10}}) + (f_{01} - \cancel{f_{10}})$$

$$= 2 f_{01} = 2(1) = 2.$$

(assuming no wraparound)

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w/ wkpaiomd $2+\frac{k}{2^k} \leq 3.$

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