SAT is NP-complete.

New problem: \(3\text{-SAT} \equiv SAT\) w/ exactly 3 literals per clause

e.g. \((x_1 \lor x_2 \lor x_3) \land (\overline{x}_1 \lor x_4 \lor \overline{x}_5) \land (x_1 \lor x_3 \lor \overline{x}_4) \land (x_2 \lor \overline{x}_3 \lor \overline{x}_6)\)

\(n = \#\) vars
\(m = \#\) clauses

\(3\text{-SAT}\) is a special case of \(\text{SAT}\).

1-SAT
\[x_1 \land x_2 \land \overline{x}_3 \land x_4 \land x_5\]
\[x_1 \land \overline{x}_1\]
easy

2-SAT \(O(n+m)\) time

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3-SAT is NP-complete

pf

1) \(3\text{-SAT} \in NP\) \(\checkmark\) (follows because \(\text{SAT} \in NP\))

2) Choose a known NP-complete problem to reduce from

Choose \(\text{SAT}\)

3) Give an \(f\) that converts \(\text{SAT}\) inputs to

\(3\text{-SAT}\) inputs s.t. satisfiability is preserved.
SAT $\xrightarrow{f} \text{3-SAT}$

Describe $f$:

- Convert each SAT clause to a set of 3-SAT clauses.
- Let $k = \#$ literals in a clause.

If $k = 1$

$$x_1 \xrightarrow{f} (x_1 \lor x_1 \lor x_1)$$

If $k = 2$

$$(x_1 \lor x_2) \xrightarrow{f} (x_1 \lor x_2 \lor x_2)$$

If $k = 3$

$$(x_1 \lor x_2 \lor x_3) \xrightarrow{f} (x_1 \lor x_2 \lor x_3)$$
k = 5

If \( \phi \) is true, then at least one literal is true. Use \( y_i \)'s to sat. remaining clauses

If \( \phi \) is false, \( y_1 \lor (\neg y_1 \lor y_2) \lor y_2 \) is false

k > 5

(\( x_1 \lor x_2 \lor \cdots \lor x_r \))

If \( k = 4 \)

In
\( x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5 \)

Set of clauses

\( \neg (x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5) \)

If \( \phi \) is true, can be extended to a setting that makes \( \phi' \) true

- a setting of \( x_\ell \) and \( x_i \)
- the two make \( \phi' \) true

- a setting of \( x_\ell \) and \( x_i \) makes \( \phi' \) false, cannot be extended to a setting that makes \( \phi' \) true

\[ \begin{align*}
\text{e.g.} & \quad x_1 = T, x_2 = x_3 = x_4 = F \Rightarrow y_1 = F \\
& \quad x_1 = x_2 = F, x_3 = x_4 = T \Rightarrow y_1 = T
\end{align*} \]

If \( \phi \) is true, then at least one literal is true, set \( y_1 \) so that the clause not containing \( x_i \) is true.

If \( \phi \) is false, then \( x_1 = x_2 = x_3 = x_4 = F \), so \( (x_1 \lor x_2 \lor y_1 \lor (x_1 \lor x_2 \lor y_1)) \) is equiv. to \( y_1 \lor \neg y_1 \), which is false for any setting of \( y_1 \).
- Described $f$.
- $f$ is poly time

clause w/ $k$ vars $\implies$ $k-2$ clauses of 3 vars.

Clauses blow up by a factor of $n$

vars blow up by a factor of $n$

- we argued that $x$ is a yes instance to SAT

\[ \implies f(x) \implies \exists \text{ 3-SAT} \]

\[ SAT \leq \text{2-SAT} \]

\[ (x_1 \lor x_2 \lor x_3 \lor x_4) \implies (x_1 \lor y_1) \land (\overline{y_1} \lor x_2) \]

Can't do this reduction.

This reduction doesn't work.

\[ \underline{2 \cdot \text{SAT} \leq 3 \cdot \text{SAT}} \]

\[ (x_1 \lor x_2) \implies (x_1 \lor x_2 \lor x_3) \]

tells me nothing
SAT ≤ 3-SAT

**Clique**

Def. a k-clique is a set of k vertices with all \((\binom{k}{2})\) edges between them.

1 2 3 4 5

Given \(G=(V, E)\) and an int \(k\).

Does \(G\) have a set of \(k\) vertices that form a k-clique?

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**G** has a 4-clique
no 5-clique

Clique is NP-complete

1) Clique ∈ NP

2) Reduce from 3-SAT
Describe:\{3-SAT instances\} $\rightarrow$ Graph $G$: $x_i$ is sat $(\Rightarrow)$

$\emptyset = (x_1 v \overline{x}_2 v x_3) \land (\overline{x}_1 v x_2 v x_3) \land (x_1 v x_2 v \overline{x}_3)$

Strategy:
- node for each appearance of a var literal
- edges between literals that can simultaneously be true
- $k = \# clauses$

If $\emptyset$ is sat, iff $G$ has a $k$-clique

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If $\emptyset$ is satisfiable, then there is a setting of the variables w/ at least one literal per clause
Set to true. This set of literals cannot contain both $x_i$ and $\overline{x}_i$ so in the graph the correspond nodes form a $k$-clique

$\leq$ If $G$ has a $k$-clique, the clique must consist of $k$ nodes, 1 per clause, and any $x_i$ or $\overline{x}_i$. Therefore you can set these literals to true and satisfy $\emptyset$.  

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