SAT is NP-complete.

New problem: 3-SAT $\equiv$ SAT w/ exactly 3 literals per clause

e.g.

$$(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_4 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee \bar{x}_4) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_5)$$

n = # vars

m = # clauses

3-SAT is a special case of SAT.

1-SAT

$x_1 \wedge x_2 \wedge \bar{x}_3 \wedge x_4 \wedge x_7$

$x_1 \wedge \bar{x}_1$

easy

2-SAT $O(n+m)$ time

Dec 1-4:07 PM

3-SAT is NP-complete

Pf

1) 3-SAT $\in$ NP ✓ (follows because SAT $\in$ NP)

2) Choose a known NP-complete problem to reduce from

Choose SAT

3) Give an f that converts SAT inputs to 3-SAT inputs s.t. satisfiability is preserved.

Dec 1-4:19 PM

$$\text{SAT} \xrightarrow{f} \text{3-SAT}$$

Describe f:

$(\quad) \wedge (\quad) \wedge$

$(\quad) \wedge (\quad) \wedge$

convert each SAT clause to a set of 3-SAT clauses

let k = # literals in a clause

Dec 1-4:22 PM

if $k=1$

$$x_1 \xrightarrow{f} (x_1 \vee x_1 \vee x_1)$$

if $k=2$

$$(x_1 \vee x_2) \xrightarrow{f} (x_1 \vee x_2 \vee x_2)$$

if $k=3$

$$(x_1 \vee x_2 \vee x_3) \xrightarrow{f} (x_1 \vee x_2 \vee x_3)$$

Dec 1-4:25 PM

if $k=4$

$\phi \equiv (x_1 \vee x_2 \vee x_3 \vee x_4) \xrightarrow{f}$ set of clauses

~~$(x_1 \vee x_3 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$~~

$\phi' \equiv (x_1 \vee x_2 \vee y_1) \wedge (x_3 \vee x_4 \vee \bar{y}_1)$

- a setting of the vars that makes ϕ true can be extended to a setting that makes ϕ' true
- a setting of the vars the makes ϕ false, cannot be extended to a setting that makes ϕ' true

e.g. $x_1 = T$ $x_2 = x_3 = x_4 = F \Rightarrow y_1 = F$

$x_1 = x_2 = F$ $x_3 = x_4 = T \Rightarrow y_1 = T$

If ϕ is true then at least one literal is true, set y_1 so that the clause not containing x' is true.

If ϕ is false, then $x_1 = x_2 = x_3 = x_4 = F$, so $(x_1 \vee x_2 \vee y_1) \wedge (x_1 \vee x_2 \vee \bar{y}_1)$ is equiv. to $y_1 \wedge \bar{y}_1$ which is false for any setting of y_1

Dec 1-4:27 PM

$k=5$

$\phi\ (x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5) \xrightarrow{f} (x_1 \vee x_2 \vee y_1) \wedge (\bar{y}_1 \vee x_3 \vee y_2) \wedge (\bar{y}_2 \vee x_4 \vee x_5)$

if ϕ is true then at least one lit. is true, use y's to sat. remaing clauses

if ϕ is false $y_1 \wedge (\bar{y}_1 \vee y_2) \wedge \bar{y}_2$ is false

$k > 5$

$(x_1 \vee x_2 \cdots \vee x_5) \to (x_1 \vee x_2 \vee y_1) \wedge (\bar{y}_1 \vee x_3 \vee y_2) \wedge (\bar{y}_2 \vee x_4 \vee y_3) \wedge (\bar{y}_3 \vee x_5 \vee y_4)$

$\vdots$

$\wedge (\bar{y}_{k-3} \vee x_{k-1} \vee x_k)$

Dec 1-4:42 PM

- Described f.
- f is poly time

clause w/ k vars $\rightarrow$ k-2 clauses of 3 vars. each.

clauses blow up by a factor of n
vars blow up by a factor of n

- we argued that x is a yes instance to SAT $\iff$ f(x) " " " " 3-SAT.

Dec 1-4:49 PM

SAT $\leq$ 2-SAT

$(x_1 \vee x_2 \vee x_3 \vee x_4) \rightarrow (x_1 \vee y_1) \wedge (\bar{y}_1 \vee x_2)$

~~can't do this reduction.~~
this reduction doesn't work.

2-SAT $\leq$ 3-SAT

$(x_1 \vee x_2) \xrightarrow{f} (x_1 \vee x_2 \vee x_2)$

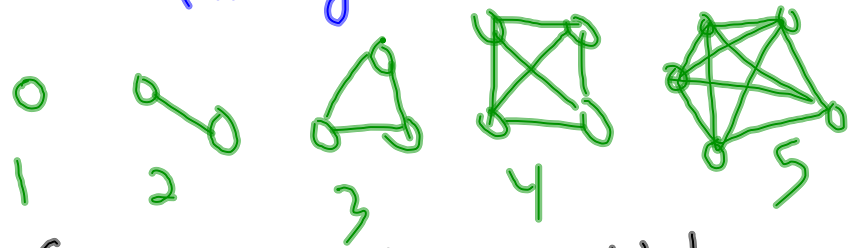
tells me nothing

Dec 1-4:52 PM

SAT $\leq$ 3-SAT

Clique

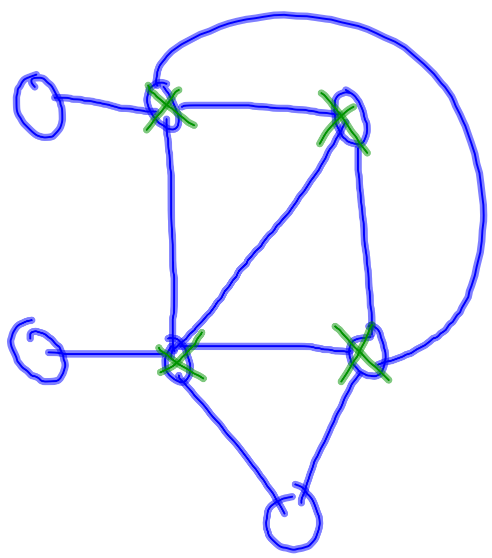
Def a k-clique is a set of k vertices with all $\binom{k}{2}$ edges between them.



Clique Given $G=(V,E)$ and an int k.

Does G have a set of k vertices that form a k-clique

Dec 1-4:57 PM



G has a 4-clique

no 5-clique

Clique is NP-complete

1) Clique $\in$ NP

2) Reduce from 3-SAT

Dec 1-5:01 PM

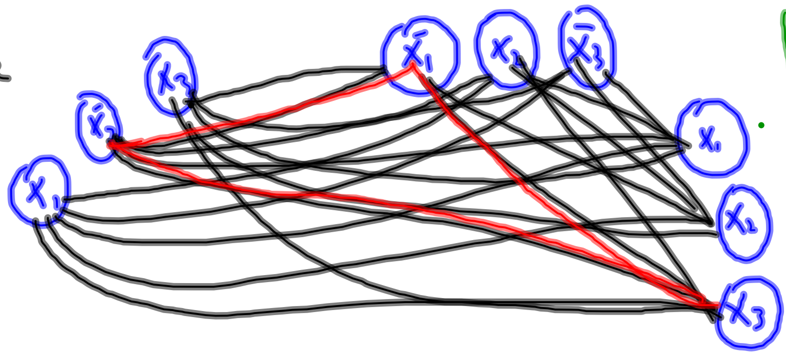
Describe $f: \{\text{3-SAT instances } x\} \to \text{Graphs } g, k$ s.t. x is sat $\iff$ g has a k-clique

$$\Phi = (\overset{T}{x_1} \vee \overset{F}{\bar{x}_2} \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \overset{T}{\bar{x}_3}) \wedge (\overset{T}{x_1} \vee x_2 \vee \overset{F}{x_3})$$

strat: - node for each appearance of a var. literal
- edges between literals that can simultaneously be true in diff. clauses

Φ is sat. iff G has a k-clique

$k = \#\text{clauses} = 3$



Dec 1-5:05 PM

⇒ If Φ is satisfiable, then there is a setting of the variables w/ at least one literal per clause set to true. This set of literals cannot contain both x_i & $\bar{x}_i$ so in the graph the corresp. nodes form a k-clique

⇐ If G has a k-clique, the clique must consist of k nodes, 1 per clause and w/o any x_i & $\bar{x}_i$ $\forall i$. Therefore you can set these literals to true and satisfy Φ.

Dec 1-5:15 PM