SAT  
3-SAT  
Clique

\[ 3 \text{-SAT} \leq \text{Clique} \]

\[ (\land (\land (\land (\land (\land (\land \text{x is sat (x)} \land \text{a k-clique} (x)) \land \text{a k-clique} (x)) \land \text{a k-clique} (x)) \land \text{a k-clique} (x)) \land \text{a k-clique} (x)) \land \text{a k-clique} (x) \land \text{a k-clique} \]

Informal: clique, whose nodes are in groups of 3 if no intragroup edges 1 is hard.

Proof shows that a "special case" of clique is NP-complete leads that clique is NP-complete.
**Vertex Cover**

Given a graph $G = (V, E)$ and an integer $k$, a vertex cover $V' \subseteq V$ is a subset of the vertices such that $\forall (u, v) \in E$, $u \in V'$ or $v \in V'$ or both. Is there a vertex cover $V'$ with $|V'| \leq k$?

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**VC in NP-complete**

**Proof**

1) $VC \in NP$

What is a VC for a clique?

VC of a $k$-clique has $k-1$ vertices
Claim: $G$ has a $k$-clique iff $G'$ has a vertex cover of size $k'$.

Proof:

$\Rightarrow$ Let $C$ be a clique in $G$. Then $C$ in $G'$ has no edges in it.

$\Rightarrow$ $V - C$ is a vertex cover of $G'$

$\Leftarrow$ Let $G'$ have a vertex cover of size $k'$. Then $V - D$ is a subset of vertices with no edges in it.

$\Leftarrow$ In $G'$, $V - D$ is a clique of size $k' = |V| + |D| - |V|. k' = k$.
**Subset Sum**

Given a set of integers \( S = \{ s_1, s_2, \ldots, s_n \} \) and a target integer \( t \).

Is there a subset \( S' \subseteq S \) such that

\[
\sum_{i \in S'} s_i = t.
\]

\( \{1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344\} \)

\( t = 3754 \quad SS \in NP \)

\( S' = \{1, 16, 64, 256, 1040, 1093, 1284\} \)

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**VC \leq SS**

Main idea:

- Think of rows as (binary) numbers, sum them, interpret the sum as problems.

1. Coins
2. What is \( t \)?
3. What about \( k \)?

\[ \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 2 \\
\end{array} \]

\( VC \) is a set of rows, s.t. each column has at least one 1.

\( VC \) is a set of row vectors, the sum shall have all non-0 components.
Problems

1) Carries
   - use base 4
2) Target sum
   - introduce "dummy" entries
3) k
   - add a column to count

G has a VC of size k
\( \subseteq \)
SS has a subset sum to \( t \)

<table>
<thead>
<tr>
<th>vert</th>
<th>( e_4 )</th>
<th>( e_3 )</th>
<th>( e_2 )</th>
<th>( e_1 )</th>
<th>( e_0 )</th>
<th>base 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1041</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1284</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1344</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1044</td>
</tr>
<tr>
<td>( y_0 )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1093</td>
</tr>
<tr>
<td>( y_1 )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>64</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>256</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3754</td>
</tr>
</tbody>
</table>

\( t = (3) \) 2 2 2 2 2 2
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\[ \Rightarrow \text{take the vertices of the vertex cover, of size } k \]

They sum to

\[ b_{xy} \Rightarrow (k) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \]

adding

\[ y \text{'s corresponding to } 1 \text{'s} \]

yielding

\[ (k) 2 2 2 2 2 \]

and this is t.

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\[ (k=) SS \text{ instance is yes } \Rightarrow G \text{ has a VC of size } k. \]

\[ \text{PF} \]

- SS is yes, set of rows in the sum
  - must include \( k \times \) rows (\( k \) vertices)
    - for each column, I
    - must choose at least one \(
      \text{that has a 1 in the col.} \Rightarrow \text{these vertices cover all the edges.} \]
Hamiltonian Cycle

Given a graph \( G = (V,E) \)
is there a cycle visiting each vertex exactly once.

Traveling Salesman Problem

Given a graph \( G = (V,E) \) w/ edge weights \( w \), int \( B \). Is there a Hamiltonian cycle \( C \) s.t.

\[ \sum_{e \in C} w_e \leq B \]