Divide & Conquer

1. Divide a problem into piece
2. Recursively solve each piece
3. Combine the solutions to the pieces

e.g. Strassen $n \times n$

1. Divided into 7 $\frac{n}{2} \times \frac{n}{2}$ problems
2. R.S.
3. Combined via 18 additions

$T(n)$ MergeSort(A, p, r) $A[p..r]$

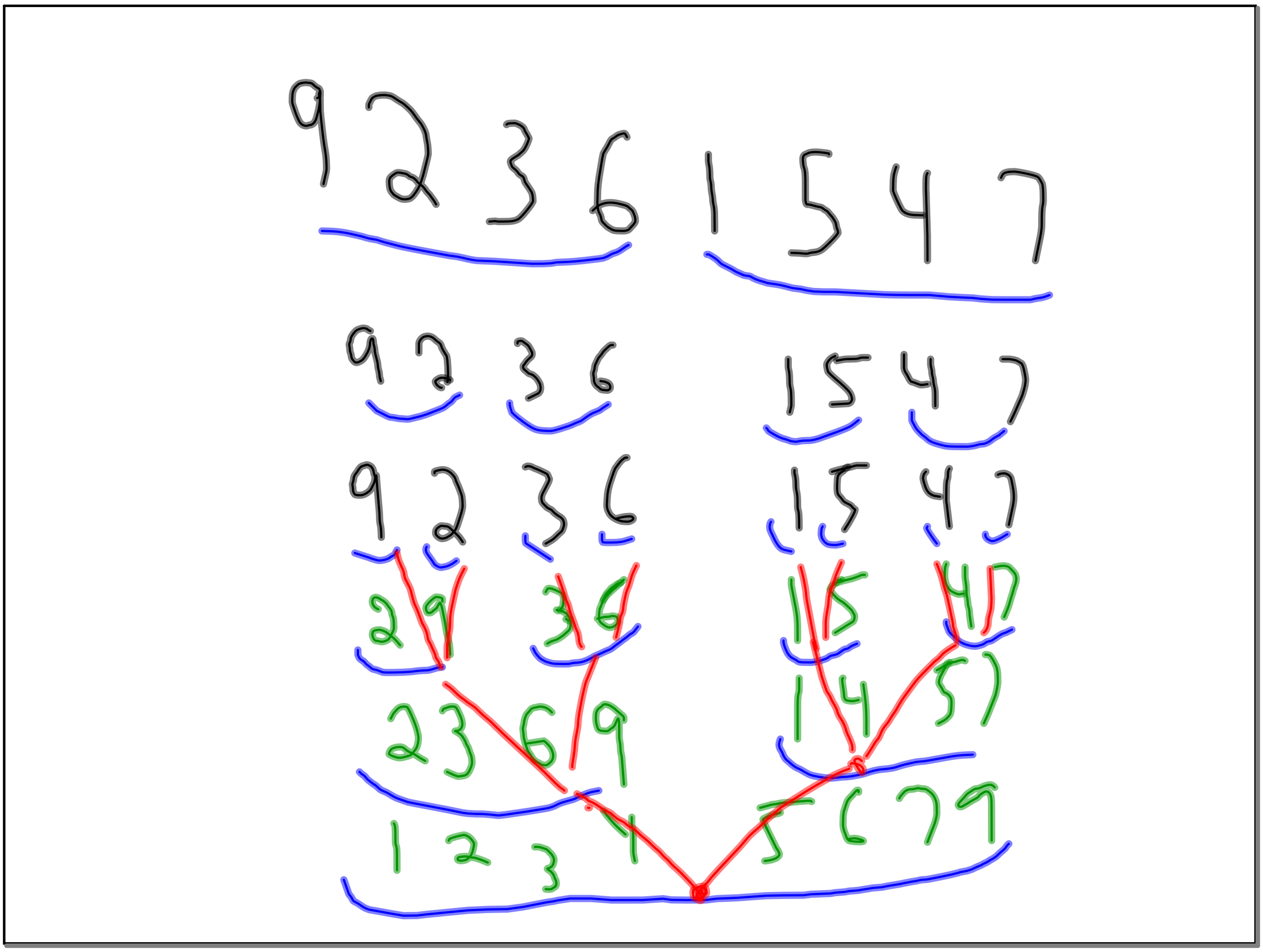
if $(p < r)$

$q = \left\lfloor \frac{p+r}{2} \right\rfloor$ 1

MergeSort(A, p, q) $T(\frac{n}{2})$

MergeSort$(A, q+1, r)$ $T(\frac{n}{2})$

Merge(A, p, q, r) $O(n)$



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Let $T(n)$ be the running time of M.S. on n items. Then

$$\boxed{T(n) = \begin{cases} 1 & \text{if } n=1 \\ 2T(\frac{n}{2})+O(n) & \text{if } n>1 \end{cases}}$$

Want a big O solution $T(n)=O(n\lg n)$.

allow "sloppiness" - not worrying about base-case, floors, ceilings.

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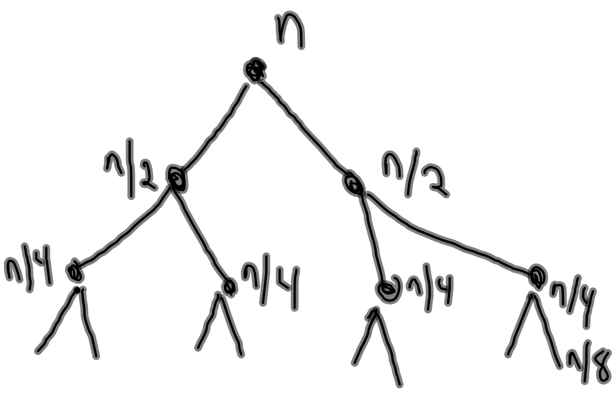
$T(n) = 2T(n/2) + n$



level		Work
0		n
1		$2\left(\frac{n}{2}\right) = n$
2		$4\left(\frac{n}{4}\right) = n$
i	$n/2^i$	$2^i\left(\frac{n}{2^i}\right) = n$
$\lg n$	1	$2^{\lg n}\left(\frac{n}{2^{\lg n}}\right) = n$

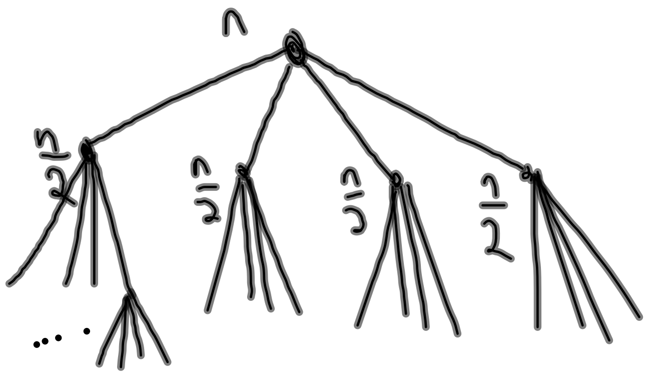
$$\text{Total Work} = \sum_{i=0}^{\lg n} n = n(\lg n + 1) = O(n \lg n).$$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

Level		Work
0	n	n
1	$n/2$	$n/2$
2	$n/4$	$n/4$
3	$n/8$	$\vdots$
$\vdots$	$\vdots$	
i	$n/2^i$	$n/2^i$
		$\vdots$
$\lg n$	$n/2^{\lg n} = 1$	1

$$\text{Total Work} = n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots + 1$$
$$= n\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$$
$$\leq 2n = O(n).$$

$$T(n) = \underline{4}\,T\left(\frac{n}{\underline{2}}\right) + \underline{\underline{n}}$$



Level		Work
0	n	n
1	$\frac{n}{2}$	$4\left(\frac{n}{2}\right) = 2n$
2	$\frac{n}{4}$	$16\left(\frac{n}{4}\right) = 4n$
3	$\frac{n}{8}$	
i	$\frac{n}{2^i}$	$4^i\left(\frac{n}{2^i}\right) = 2^i \cdot n$
$\lg n$	$\frac{n}{2^i} = 1$	$n^2(1) = n^2$

$\#\text{of leaves} = 4^{\lg n} = n^2$

$$\text{Total work} = n + 2n + 4n + 8n + \cdots \frac{n^2}{2} + n^2$$
$$= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \cdots + 4n + 2n + n$$
$$= n^2\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$$
$$\le 2n^2 = O(n^2)$$

Informal MT

$$T(n) = af\left(\frac{n}{b}\right) + f(n)$$

compare

$f(n)$ $\quad n^{\lg_b a}$

$<$ $\quad O(n^{\lg_b a})$

$>$ $\quad O(f(n))$

$=$ $\quad O(f(n)\lg n)$

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e.g. $T(n) = 2T(\frac{n}{2}) + n$

$n^{\lg_2 2}$ $\quad n$

$n \quad n \quad O(n \lg n)$

$T(n) = T(n/2) + n$

$n^{\lg_2 1} \quad n$

$1 \quad n \quad O(n)$

$T(n) = 4T(\frac{n}{2}) + n$

$n^{\lg_2 4} \quad n \quad O(n^2)$

$n^2 \quad n$

$$T(n) = 7T\left(\frac{n}{2}\right) + n^2$$

$$n^{\lg_2 7} \qquad n^2 \qquad O(n^{\lg_2 7})$$

$$T(n) = 4T\left(\frac{n}{3}\right) + n^{1.5}$$

$$n^{\lg_3 4} \qquad n^{1.5} \qquad O(n^{1.5})$$

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$n^{\lg_2 1} \qquad 1 \qquad O(\lg n$$

$$1 \qquad 1$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n\lg n$$

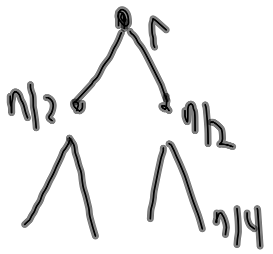
$$n^{\lg_2 2} \quad n\lg n$$

$$n \quad n\lg n$$

$$n < n\lg n < n^{1+\varepsilon}$$ MT does not apply.

$$T(n) = 2T(n-1) + 1 \qquad T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + n$$

$$T(n) = 2T\left(\tfrac{n}{2}\right) + n\lg n$$



level		work
0		$n\lg n$
1		$2\left(\frac{n}{2}\lg\left(\frac{n}{2}\right)\right)$
2		$4\left(\frac{n}{4}\lg\left(\frac{n}{4}\right)\right)$
i	$n/2^i$	$2^i\left(\frac{n}{2^i}\lg\left(\frac{n}{2^i}\right)\right)$
$\lg n - 1$		$\frac{n}{2}(2\ \lg 2)$

$$\text{Total} = \sum_{i=0}^{\lg n} 2^i\left(\frac{n}{2^i}\lg\left(\frac{n}{2^i}\right)\right)$$

$$= \sum_{i=0}^{\lg n} n(\lg n - \lg 2^i)$$

$$= \sum_{i=0}^{\lg n} n(\lg n - i)$$

$$= \sum_{i=0}^{\lg n} n\lg n - n\sum_{i=0}^{\lg n} i$$

$$= n\lg^2 n - \frac{\lg n(\lg n + 1)}{2} n$$

$$= n\lg^2 n - \frac{n\lg^2 n}{2} - \frac{n\lg n}{2}$$

$$= \frac{n\lg^2 n}{2} - \frac{n\lg n}{2}$$

$$n\lg^2 n\left(\frac{1}{2} - \frac{1}{2\lg n}\right) = O(n\lg^2 n)$$

Method 2 Guess & check.

$$T(n) = 3T\left(\frac{n}{3}\right) + n \qquad n^{\lg_3 3} \quad n \quad n\lg n$$

Guess $T(n) = O(n\lg n)$

Guess $T(n) \le cn\lg n$ for some $c > 0$

Assume $T(n') \le cn'\lg n'$ for $n' < n$

Prove for n

$$T(n) = 3T\left(\frac{n}{3}\right) + n$$

$$\le 3\left(c\frac{n}{3}\lg\left(\frac{n}{3}\right)\right) + n$$

$$= cn(\lg n - \lg 3) + n$$

$$= cn\lg n + n - c\lg 3\, n$$

Choose c so that

$1 - c\lg 3 < 0$

$c > 1/\lg 3$

Choose $c = 1$

$n - \lg 3 n < 0$

$< \le cn\lg n$ ⊗

$\vdots$

$cn\lg n + cn$

can't choose $c>0$ s.t.

$cn\lg n + cn \le cn\lg n$

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