Indicator Random Variables

A be an event

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{o.w.} \end{cases}$$

Q. What is the expected # of heads when I flip 1 coin?

let Y be a r.v. that denotes heads or tails.

$$X_H = I\{Y \text{ is heads}\} = \begin{cases} 1 & \text{if } Y \text{ is heads (H)} \\ 0 & \text{o.w. (T)} \end{cases}$$

$$E[X_H] = \underbrace{Pr(X_H=1)\cdot 1}_{X_H = 1} + \underbrace{Pr\{X_H=0\}\cdot 0}_{X_H = 0}$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{2}$$

Sep 29-4:08 PM

Linearity of expectation:

Let X & Y be 2 random variables

$$E[X+Y] = E[X] + E[Y].$$

Sep 29-4:18 PM

n coin flips

$E[\#\text{heads}]$

events 0H, 1H, 2H, 3H, ... nH

$$= \sum_{i=0}^{n} \Pr(i \text{ heads in } n \text{ flips}) \cdot i$$

$\vdots$

alt.: "factor" by whether heads comes up on flip i

Sep 29-4:23 PM

let X_j = # of heads on flip j

$(I\{i \text{ flip is heads}\})$

Let X = total # of heads on n flips

$$X = \sum_{j=1}^{n} X_j$$

$$E[X] = E\left[\sum_{j=1}^{n} X_j\right] = \sum_{j=1}^{n} E[X_j]$$

$$= \sum_{j=1}^{n} \frac{1}{2} = n/2 \;//$$

Sep 29-4:27 PM

Hiring

Let X_j be the # of people hired when person j is interviewed
(IRV, 0 or 1)

X = total # of people hired

$$X = \sum_{j=1}^{n} X_j$$

$$E[X] = E\left[\sum_{j=1}^{n} X_j\right] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} \Pr(X_j = 1)$$

$$E[X_j] = \Pr(X_j=1)\cdot 1 + \cancel{\Pr(X_j=0)}\cdot 0$$
$$= \Pr(X_j = 1)$$

Sep 29-4:31 PM

What is $\Pr(X_j=1)$, prob. that we hire on the j^{th} day.

$$\Pr(X_1=1) = 1$$
$$\Pr(X_2=1) = \frac{1}{2}$$
$$\Pr(X_3=1) = \frac{1}{3}$$

Person 3 > Person 1
Person 2

$$\Pr(X_j=1) = \frac{1}{j} \quad //$$

$$E[X] = \sum_{j=1}^{n} \Pr(X_j=1) = \sum_{j=1}^{n} \frac{1}{j} \approx \ln n.$$

Sep 29-4:35 PM

Assumption: Candidates come in a random order

probabilistic analysis

Remove Assumption: Force the candidates to come in a random order by randomly permuting the data before we start

Moving the bad case from something out of our control to something we control.

Sep 29-4:45 PM

Think about a sorting alg. whose bad case is reverse sorted order 10 9 8 7 6 5 4 3 2 1

data is random: bad case is reverse sorted

alg. is random: some particular set of coin flips w/ Pr $\frac{1}{n!}$ that makes the alg. slow.

rand(a,b) - returns a "random" int. between a & b.

Sep 29-4:49 PM

rand.pdf (application/pdf Object) - Mozilla Firefox

RANDOMIZE-IN-PLACE(A)

```
n ← length[A]
for i ← 1 to n
    do swap A[i] ↔ A[RANDOM(i, n)]
```

Lemma Procedure RANDOMIZE-IN-PLACE computes a uniform random permutation.

$(n)(n-1)(n-2)\cdots(n-k+1)$

Def Given a set of n elements, a k-permutation is a sequence containing k of the n elements.

There are $n!/(n-k)!$ possible k-permutations of n elements

Sep 29-5:01 PM

Birthday Paradox

n people

$E[\#\text{ of pairs w/ same birthday}]$

X_{ij} = irv i & j have same birthday

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

$$E(X) = E\left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

(Pr i&j have same b.d)

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365}$$

$$= \binom{n}{2} \frac{1}{365} = \frac{n(n-1)}{2 \cdot 365}$$

$n = 23$ .69
$n = 28$ 1.03
$n = 64$ 5.5
$n = 90$ 10.9

Sep 29-5:08 PM

n coins, what is the longest streak of heads

HTTTTTTHTHTHHT....

X_{ik} = i.r.v. for Heads on flips i to $i+k-1$

X_k = # of streaks of length k

$= \sum_{i=1}^{n-k+1} X_{ik}$

$$E[X_k] = \sum_{i=1}^{n-k+1} E[X_{ik}]$$

$$= \sum_{i=1}^{n-k+1} \Pr(H \text{ from } i \text{ to } i+k-1)$$

$$\sum_{i=1}^{n-k+1} \frac{1}{2^k} = \frac{n-k+1}{2^k}$$

$k=3$

$\approx \frac{n}{8}$

$k=n$

$\approx \frac{1}{2^n}$

let $k = c\lg n$

$$\approx \frac{n}{2^{c\lg n}} \approx \frac{n}{n^c} = n^{1-c}$$

$c=\frac{1}{2}$

$\sqrt{n}$

$c=5$

$\frac{1}{n^4}$

Sep 29-5:20 PM