Indicators Random Variables

A be an event

\[ I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{o.w.} \end{cases} \]

Q. What is the expected # of heads when I flip a coin?

Let \( Y \) be a r.v. that denotes heads or tails.
\[ X_H = I\{Y \text{ is heads}\} = \begin{cases} 1 & \text{if } Y \text{ is heads (H)} \\ 0 & \text{o.w.} \end{cases} \]

\[ E[X_H] = \frac{\text{Pr}\{X_H = 1\}}{\text{Pr}\{X_H = 0\}} \cdot 1 + \frac{\text{Pr}\{X_H = 0\}}{\text{Pr}\{X_H = 0\}} \cdot 0 \]

\[ = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2} \]

Linearity of expectation:

Let \( X \) and \( Y \) be 2 random variables

\[ E[X+Y] = E[X] + E[Y] \]
\[ n \text{ coin flips} \]
\[ E[\# \text{heads}] \]
\[ = \sum_{i=0}^{n} \Pr(\text{i heads in } n \text{ flips}) \cdot i \]

alt: “factor” by whether heads comes up on flip i

let \( X_j = \# \text{of heads on flip } j \)
\( (I\{ \text{i flip is heads} \}) \)

Let \( X = \text{total } \# \text{ of heads on } n \text{ flips} \)
\[ X = \sum_{j=1}^{n} X_j \]
\[ E[X] = E\left( \sum_{j=1}^{n} X_j \right) = \sum_{j=1}^{n} E[X_j] \]
\[ = \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2} \]
Hiring

Let $X_j$ be the # of people hired when person $j$ is interviewed ($\text{IRV, 0 or 1}$)

$X = \text{total # of people hired}$

$X = \sum_{j=1}^{n} X_j$

$E[X] = E[\sum_{j=1}^{n} X_j] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} \text{Pr}(X_j=1)$

$E[X_j] = \text{Pr}(X_j=1) \cdot 1 + \text{Pr}(X_j=0) \cdot 0$

$= \text{Pr}(X_j=1)$

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What is $\text{Pr}(X_j=1)$, prob. that we hire on the $j$th day

$\text{Pr}(X_1=1) = 1$

$\text{Pr}(X_2=1) = \frac{1}{2}$

$\text{Pr}(X_3=1) = \frac{1}{3}$

$\text{Pr}(X_j=1) = \frac{1}{j}$

$E[X] = \sum_{j=1}^{n} \text{Pr}(X_j=1) = \sum_{j=1}^{n} \frac{1}{j} \approx \ln n$. 

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Assumption: Candidates come in a random order.

Probabilistic analysis

Remove Assumption: Force the candidates to come in a random order by randomly permuting the data before we start.

Moving the bad case from something out of our control to something we control.

Think about a sorting alg. whose bad case is reverse sorted order 10987654321.

Data is random: bad case is reverse sorted.

Alg. is random: some particular set of coin flips w/ Pr \( \frac{1}{n^2} \) that makes the alg. slow.

\( \text{rand}(a,b) \) - returns a "random" int. between \( a \) and \( b \).
**RANDOMIZE-IN-PLACE(A)**

1. \( n \leftarrow \text{length}[A] \)
2. for \( i \leftarrow 1 \) to \( n \)
3. do swap \( A[i] \leftarrow A[\text{RANDOM}(i, n)] \)

**Lemma**  Procedure **RANDOMIZE-IN-PLACE** computes a uniform random permutation.

**Def**  Given a set of \( n \) elements, a \( k \)-permutation is a sequence containing \( k \) of the \( n \) elements.

There are \( n! / (n-k)! \) possible \( k \)-permutations of \( n \) elements.

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**Birthday Paradox**

- \( n \) people

\[ E[\text{# of pairs with same birthday}] \]

\[ X_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ have same birthday} \\ 0 & \text{otherwise} \end{cases} \]

\[ X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ E(X) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \\ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365} \]

\[ = \frac{n(n-1)}{2} \times \frac{1}{365} \]

For \( n = 23 \):

- \( 0.50 \)

- \( 0.69 \)

- \( 0.79 \)

- \( 0.89 \)

- \( 0.90 \)

- \( 10.9 \)
n Coins, what's the longest streak of heads

HHTTTTTT HTTTHHHTT . . .

\[ X_{lk} = \text{i.i.d. for Headson flips} \]
\[ X_k = \# \text{of heads of length } k \]
\[ = \sum_{i=1}^{n-k+1} X_{ik} \]

\[ E[X_k] = \sum_{i=1}^{n-k+1} E[X_{ik}] \]
\[ = \sum_{i=1}^{n-k+1} P(H \text{ from } i \text{ to } i+k-1) \]
\[ \frac{\sum_{i=1}^{n-k+1} \frac{k!}{a^k}}{a^k} = \frac{n-k+1}{a^k} \]

let \( k = \log n \)
\[ \frac{n}{\log n} \leq \frac{n}{n^c} = n^{-c} \]

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