Linear programs

$$\begin{aligned} &\textit{maximize} \; \textstyle\sum_{j=1}^{n} c_j x_j \\ &\textbf{subject to} \\ &\sum_{j=1}^{n} a_{ij} x_j \quad \leq \quad b_i \;\; \textbf{for} \; i = 1, 2, \ldots, m \\ &x_j \quad \geq \quad 0 \;\; \textbf{for} \; j = 1, 2, \ldots, n \; . \end{aligned}$$

Note: Any set of linear inequalities with linear objective function can be converted to this form.

Shortest Path LP

$$\begin{aligned}
&\textit{maximize} \; d_t \\
&\textbf{subject to} \\
&\quad d_v \le d_u + w(u,v) \;\; \textbf{for each edge} \; (u,v) \in E \; , \\
&\quad d_s = 0 \; .
\end{aligned}$$

Maximum Flow LP

$$\begin{aligned}
\textit{maximize} \sum_{v \in V} f_{sv} &- \sum_{v \in V} f_{vs} \\
\textbf{subject to} & \\
f_{uv} &\leq c(u,v) \ \ \textbf{for each } u, v \in V \\
\sum_{v \in V} f_{vu} &= \sum_{v \in V} f_{uv} \ \ \textbf{for each } u \in V - \{s, t\}\ , \\
f_{uv} &\geq 0 \ \ \textbf{for each } u, v \in V\ .
\end{aligned}$$

Minimum Cost Flow LP

In mininum cost flow, edges have costs in addition to capacities.

$$
\begin{array}{rcl}
\textit{minimize} & \sum_{(u,v)\in E} a(u,v) f_{uv} & \\
\textbf{subject to} & & \\
f_{uv} & \leq & c(u,v) \ \textbf{for each } u,v \in V \ , \\
\sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} & = & 0 \ \textbf{for each } u \in V - \{s,t\} \ , \\
\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & = & d \ , \\
f_{uv} & \geq & 0 \ \textbf{for each } u,v \in V \ .
\end{array}
$$

Multicommodity Flow LP

In multicommodity flow, we have a set of commdodities, each of which must be sent as a flow, while the commodities obey joint capacity constraints.

$$
\begin{aligned}
\textit{minimize} \quad & 0 \\
\textbf{subject to} \quad & \\
\sum_{i=1}^{k} f_{iuv} &\le c(u,v) \ \textbf{ for each } u, v \in V \ , \\
\sum_{v \in V} f_{iuv} - \sum_{v \in V} f_{ivu} &= 0 \ \textbf{ for each } i = 1, 2, \ldots, k \textbf{ and} \\
& \quad \textbf{for each } u \in V - \{s_i, t_i\} \ , \\
\sum_{v \in V} f_{i,s_i,v} - \sum_{v \in V} f_{i,v,s_i} &= d_i \ \textbf{ for each } i = 1, 2, \ldots, k \ , \\
f_{iuv} &\ge 0 \ \textbf{ for each } u, v \in V \textbf{ and} \\
& \quad \textbf{for each } i = 1, 2, \ldots, k \ .
\end{aligned}
$$

Duality

Primal

$$
\begin{array}{rcl}
\textit{maximize} & \sum_{j=1}^{n} c_j x_j & \\
\textbf{subject to} & & \\
\sum_{j=1}^{n} a_{ij} x_j & \leq & b_i \;\; \textbf{for } i = 1, 2, \ldots, m \\
x_j & \geq & 0 \;\; \textbf{for } j = 1, 2, \ldots, n \;\; .
\end{array}
$$

Dual

$$
\begin{array}{rcl}
\textit{minimize} & \sum_{i=1}^{m} b_i y_i & \\
\textbf{subject to} & & \\
\sum_{i=1}^{m} a_{ij} y_i & \geq & c_j \;\; \textbf{for } j = 1, 2, \ldots, n \;\; , \\
y_i & \geq & 0 \;\; \textbf{for } i = 1, 2, \ldots, m \;\; .
\end{array}
$$

LP Duality **If both the primal and dual solution are feasible (have valid solutions), then the optimal solutions have the same objective value.**