Maximum Flows

- **A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity .**
- **If $(u, v) \notin E$, we assume that $c(u, v) = 0$.**
- **We distinguish two vertices in a flow network: a source s and a sink t.**

A flow in G is a real-valued function $f : V \times V \to R$ that satisfies the following three properties:

Capacity constraint: For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$.

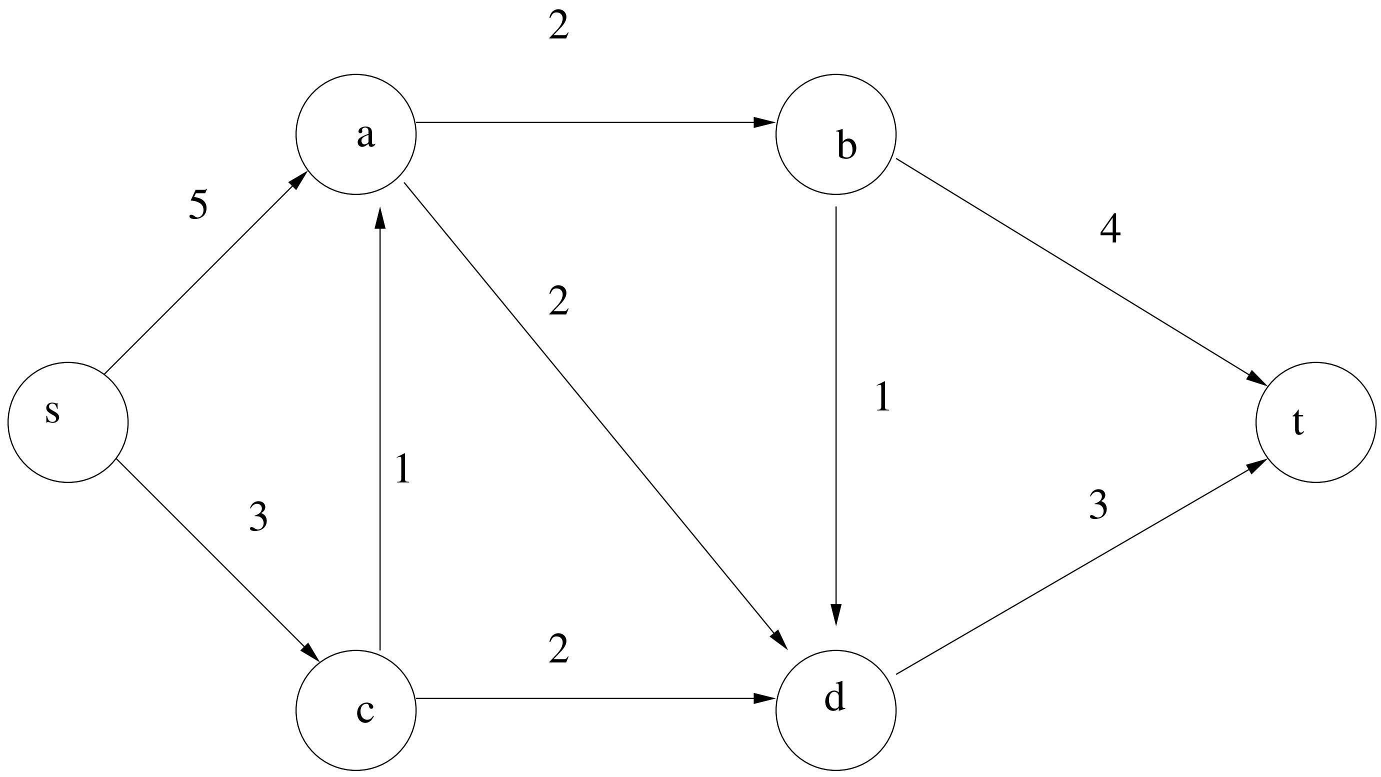
Flow conservation: For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \ .$$

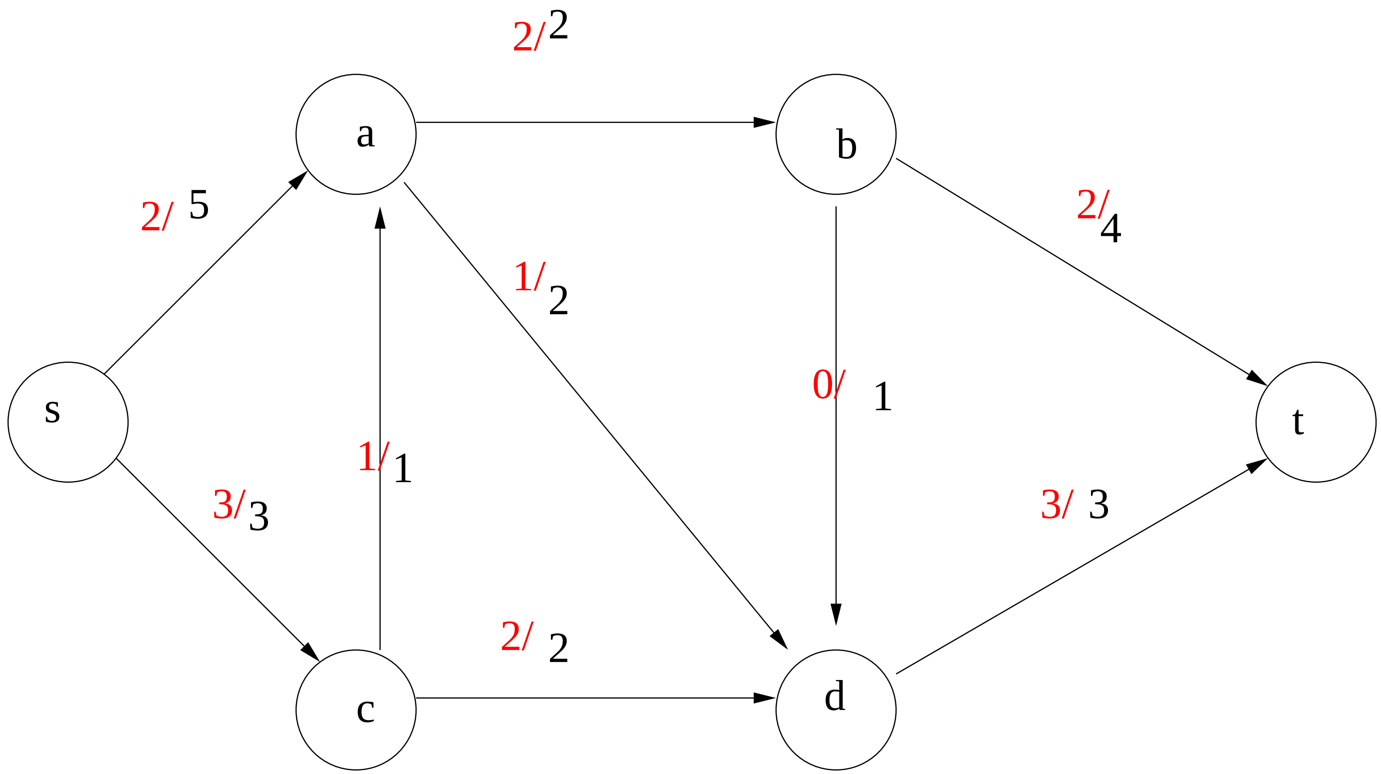
The value of a flow f is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) \ , \qquad (1)$$

Example



Solutions



Ford Fulkerson

Ford-Fulkerson-Method(G, s, t)

initialize flow f **to** 0
while there exists an augmenting path p
    augment flow f **along** p
return f

- **The residual capacity of** (u, v)**, is**

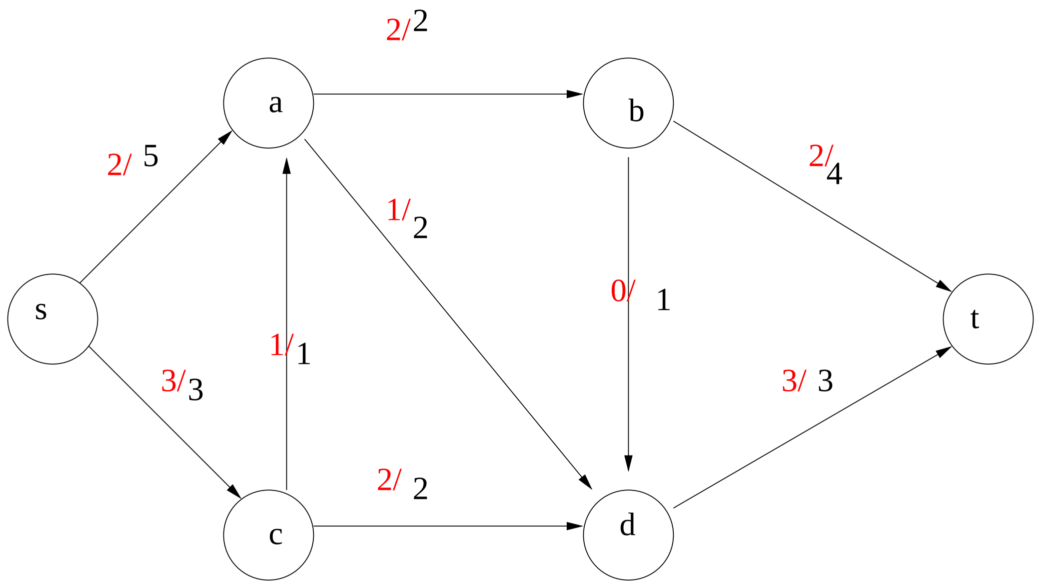
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \textbf{if } (u,v) \in E \ , \\ f(v,u) & \textbf{if } (v,u) \in E \ , \\ 0 & \textbf{otherwise} \ . \end{cases} \qquad (2)$$

- **The residual graph** G_f **is the graph consisting of edges with positive residual capacity**
- **Flows add**

s-t Cuts

An s-t cut satsfies

- $s \in S$, $t \in T$
- $S \cup T = V$, $S \cap T = \emptyset$



$$c(S,T) = \sum_{u \in S, v \in T} c(u,v)$$

$$f(S,T) = \sum_{u \in S, v \in T} f(u,v)$$

- **For all cuts (S,T) and all feasible flows f, $f(S,T) \leq c(S,T)$**
- **For all pairs of cuts (S_1,T_1) and (S_2,T_2), and all feasible flows f, $f(S_1,T_1) = f(S_2,T_2)$.**

Max-flow min-cut theorem

If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. **f is a maximum flow in G.**
2. **The residual network G_f contains no augmenting paths.**
3. **$|f| = c(S, T)$ for some cut (S, T) of G.**

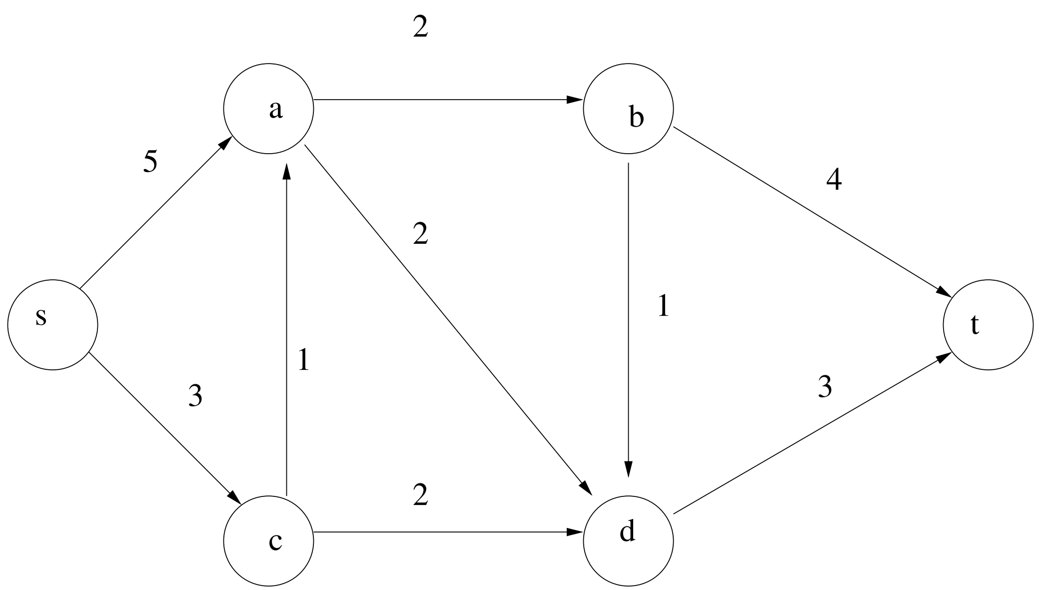
Ford Fulkerson expanded

$Ford - Fulkerson(G, s, t)$

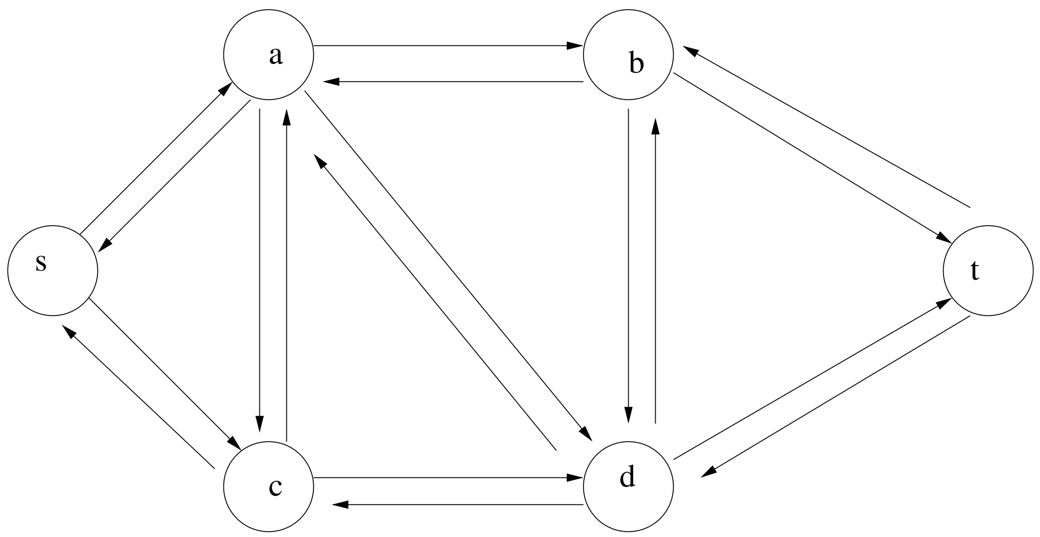
for each edge $(u, v) \in G.E$
    $(u, v).f = 0$
while there exists a path p **from** s **to** t **in the residual network** G_f
    $c_f(p) = \min\{c_f(u, v) : (u, v)$ **is in** $p\}$
    for each edge (u, v) **in** p
        if $(u, v) \in E$
            $(u, v).f = (u, v).f + c_f(p)$
        else $(v, u).f = (v, u).f - c_f(p)$

Algorithm

Graph

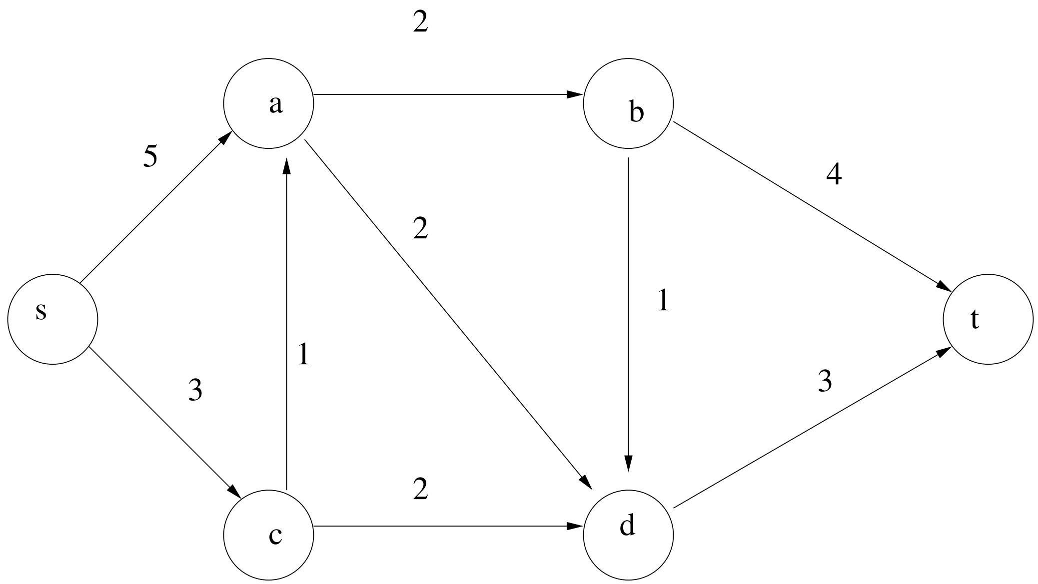


Residual graph

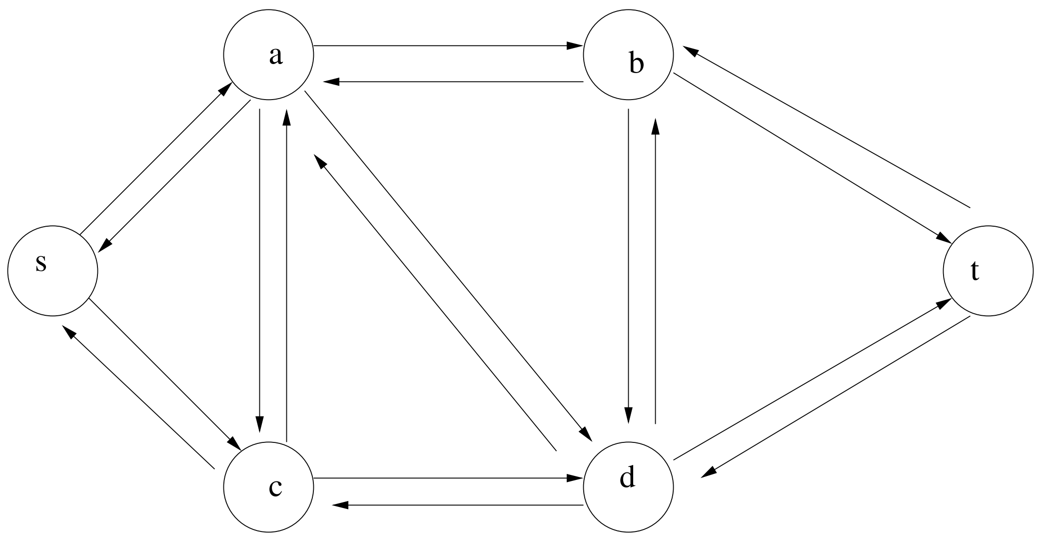


Algorithm

Graph

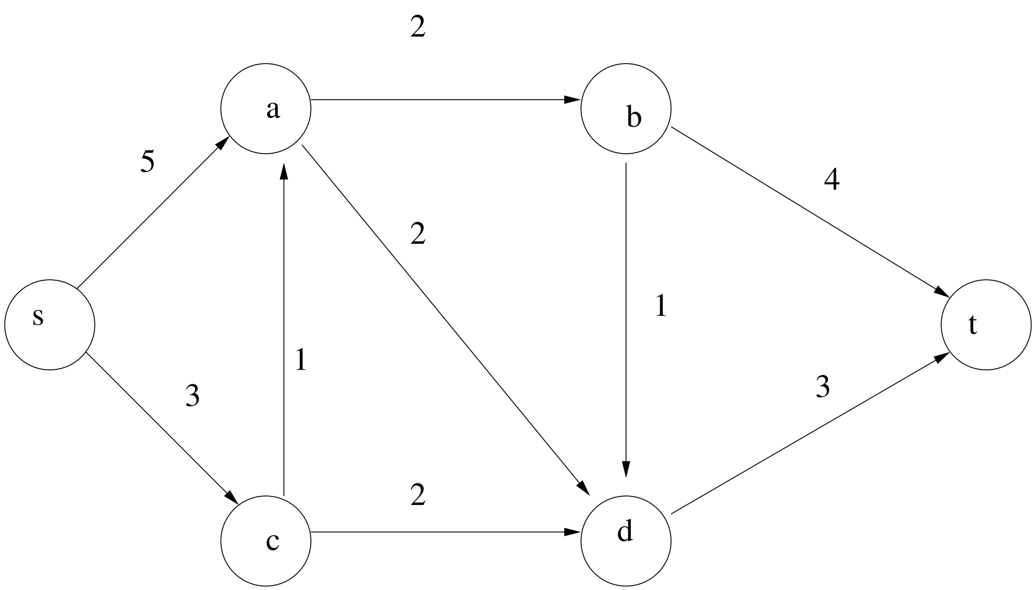


Residual graph



Algorithm

Graph



Residual graph