Greedy

Consider a set of requests for a room. Only one person can reserve the room at a time, and you want to allow the maximum number of requests. The requests for periods (s_i, f_i) are:

$$(1,4),(3,5),(0,6),(5,7),(3,8),(5,9),(6,10),(8,11),(8,12),(2,13),(12,14)$$

Which ones should we schedule?

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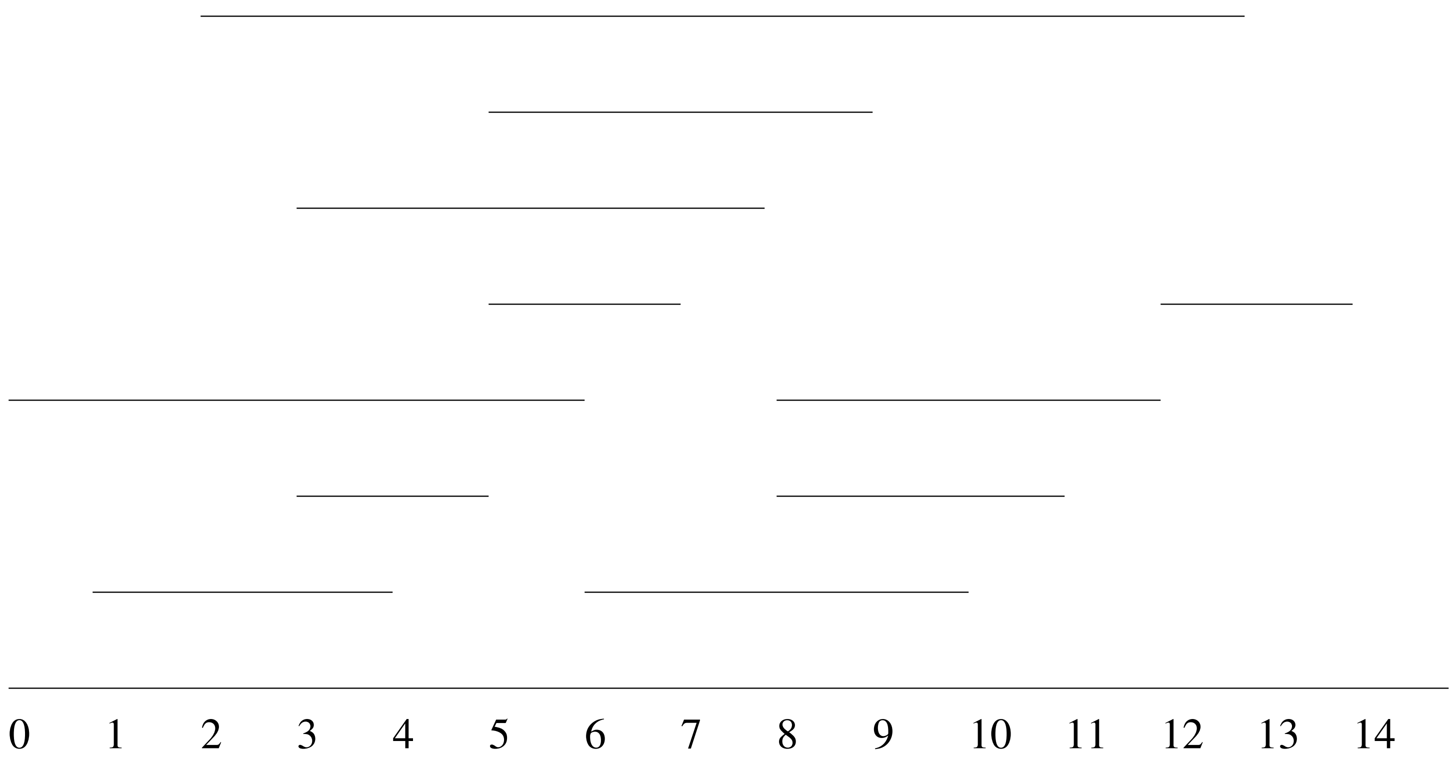

Code

```
Sort by finishing time, renumber with 1 having earliest finishing time
Output 1
last = f_1
for i = 2 to n
    if (s_i ≤ last)
        Output i
        last = f_i
```

Proving a Greedy Algorithm is Optimal

Two components:

1. **Optimal substructure**
2. **Greedy Choice Property: There exists an optimal solution that is consistent with the greedy choice made in the first step of the algorithm.**

Optimal Substructure

- **Let $c[i,j]$ be the number of activities scheduled from time i to time j**

$$c[i,j] = \begin{cases} 0 & \textbf{if } S_{ij} = \emptyset \ , \\ \max_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\} & \textbf{if } S_{ij} \neq \emptyset \end{cases} \quad . \tag{1}$$

Greedy Choice

Greedy Choice Property

1. Let S_k be a nonempty subproblem containing the set of activities that finish after activity a_k.
2. Let a_m be an activity in S_k with the earliest finish time.
3. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k.

Proof

- Let A_k be a maximum-size subset of mutually compatible activities in S_k,
- let a_j be the activity in A_k with the earliest finish time.
- If $a_j = a_m$, we are done, since we have shown that a_m is in some maximum-size subset of mutually compatible activities of S_k.
- If $a_j \neq a_m$, let the set $A'_k = A_k - \{a_j\} \cup \{a_m\}$
- The activities in A'_k are disjoint, because
  - the activities in A_k are disjoint,
  - a_j is the first activity in A_k to finish,
  - $f_m \leq f_j$.
- Since $|A'_k| = |A_k|$, we conclude that A'_k is a maximum-size subset of mutually compatible activities of S_k, and it includes a_m.

Procedure for Designing a Greedy Algorithm

1. **Identify optimal substructure**
2. **Cast the problem as a greedy algorithm with the greedy choice property**
3. **Write a simple iterative algorithm**

Robbery

- I want to rob a house and I have a knapsack which holds B pounds of stuff
- I want to fill the knapsack with the most profitable items

item	1	2	3
weight	10	20	30
value	60	100	120
value/weight	6	5	4

Two variants

- **integral knapsack**: Take an item or leave it
- **fractional knapsack**: Can take a fraction of an item (infinitely divisible)

Fractional vs. Integral Knapsack

- **Both fractional and integral knapsack have optimal substructure.**
- **Only fractional knapsack has the greedy choice property.**

Fractional Knapsack

Greedy Choice Property: Let j be the item with maximum v_i/w_i. Then there exists an optimal solution in which you take as much of item j as possible.

Proof

- Suppose fpoc, that there exists an optimal solution in you didn't take as much of item j as possible.
- If the knapsack is not full, add some more of item j, and you have a higher value solution. Contradiction
- We thus assume the knapsack is full.
- There must exist some item $k \neq j$ with $\frac{v_k}{w_k} < \frac{v_j}{w_j}$ that is in the knapsack.
- We also must have that not all of j is in the knapsack.
- We can therefore take a piece of k, with ϵ weight, out of the knapsack, and put a piece of j with ϵ weight in.
- This increases the knapsacks value by

$$\epsilon\frac{v_j}{w_j} - \epsilon\frac{v_k}{w_k} = \epsilon\left(\frac{v_j}{w_j} - \epsilon\frac{v_k}{w_k}\right) > 0$$

Contradition to the original solution being optimal.

Algorithm

1. **Sort items by v_j/w_j, renumber.**
2. **For $i = 1$ to n**
   - **Add as much of item i as possible**

Question **Why does this fail for integer knapsack.**