Shortest Paths

- **Input: weighted, directed graph $G = (V, E)$, with weight function $w : E \rightarrow$ R.**
- **The weight of path $p =< v_0, v_1, \ldots, v_k >$ is the sum of the weights of its constituent edges:**

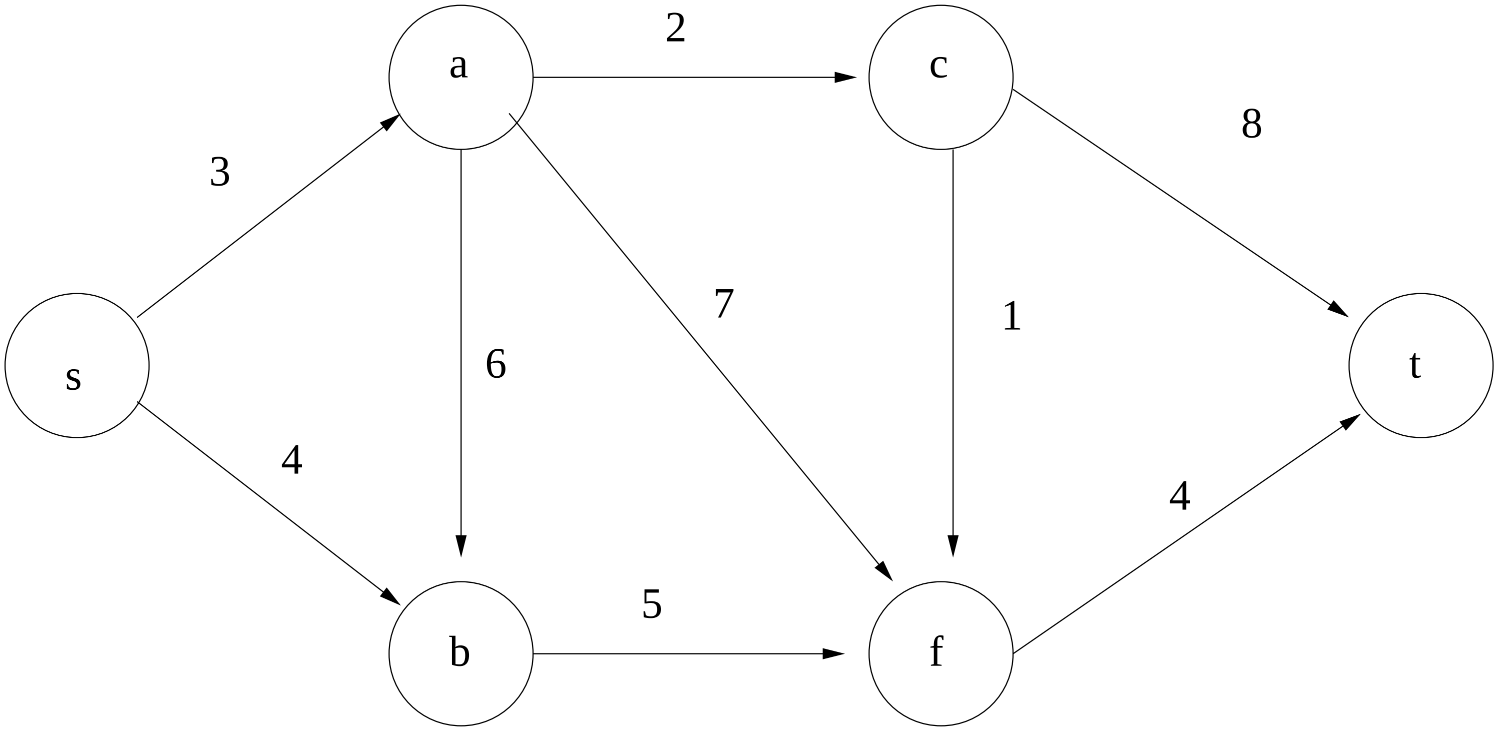
$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i) \ .$$

- **The shortest-path weight from u to v is**

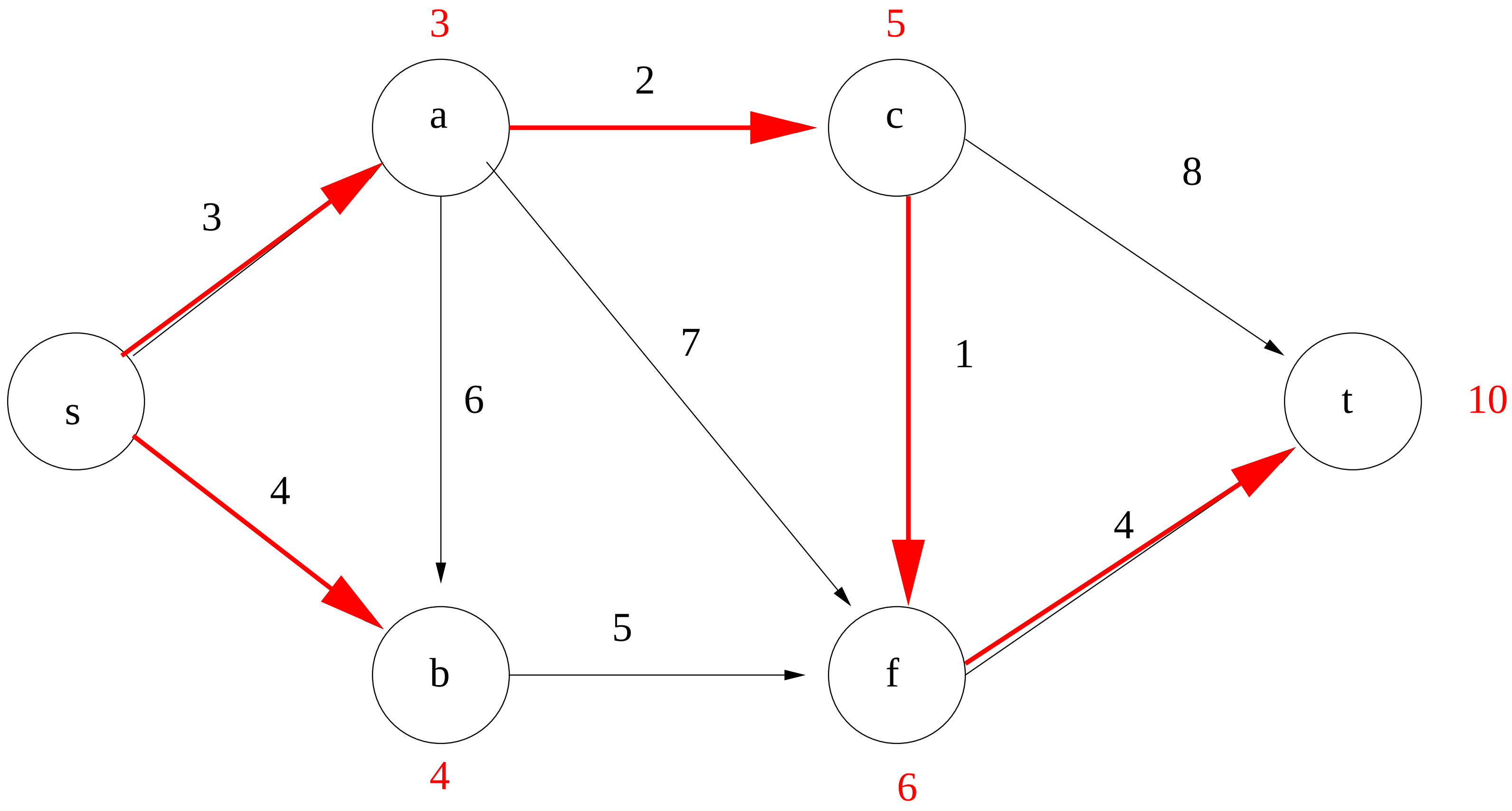
$$\delta(u, v) = \begin{cases} \min\{w(p)\} & \textbf{if there is a path } p \textbf{ from } u \textbf{ to } v \ , \\ \infty & \textbf{otherwise} \ . \end{cases}$$

- **A shortest path from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.**

Example



Solution



Shortest Paths

Shortest Path Variants

- **Single Source-Single Sink**
- **Single Source (all destinations from a source s)**
- **All Pairs**

Defs:

- **Let $\delta(v)$ be the real shortest path distance from s to v**
- **Let $d(v)$ be a value computed by an algorithm**

Edge Weights

- **All non-negative**
- **Arbitrary**

Note: **Must have no negative cost cycles**

Single Source Shortest Paths

Key Property: Subpaths of shortest paths are shortest paths Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbf{R}$, let $p =< v_1, v_2, \ldots, v_k >$ be a shortest path from vertex v_1 to vertex v_k and, for any i and j such that $1 \leq i \leq j \leq k$, let $p_{ij} =< v_i, v_{i+1}, \ldots, v_j >$ be the subpath of p from vertex v_i to vertex v_j. Then, p_{ij} is a shortest path from v_i to v_j.

Note: this is optimal substructure

Corollary 1 For all edges $(u, v) \in E$,

$$\delta(v) \leq \delta(u) + w(u, v)$$

Corollary 2 Shortest paths follow a tree of edges for which

$$\delta(v) = \delta(u) + w(u, v)$$

More precisely, any edge in a shortest path must satisfy

$$\delta(v) = \delta(u) + w(u, v)$$

Relaxation

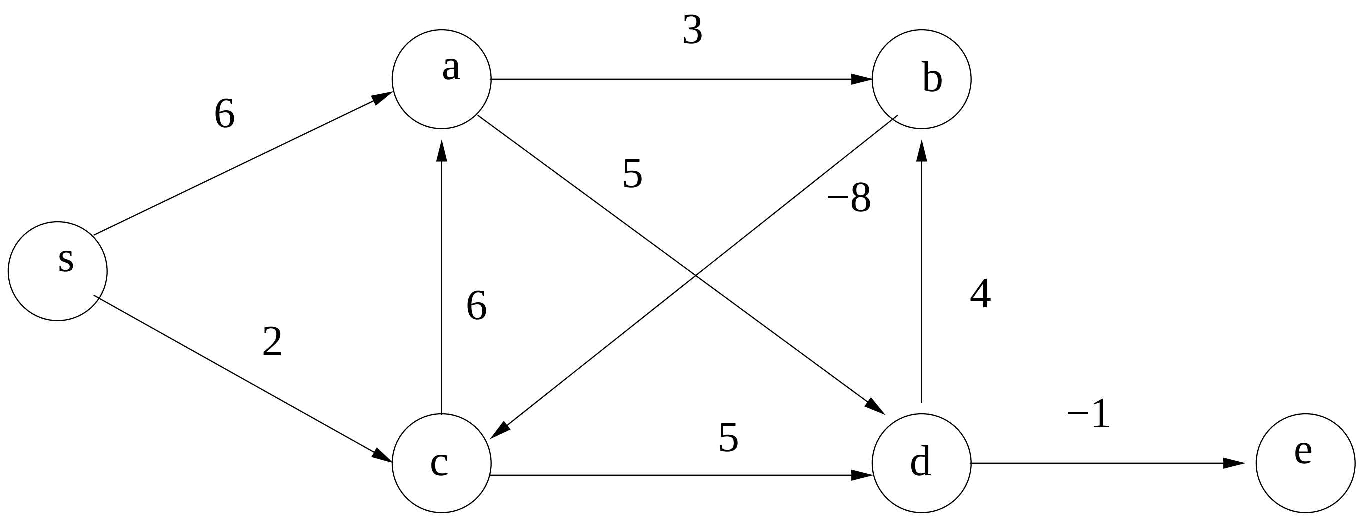
Relax(u, v, w)

```
if d[v] > d[u] + w(u, v)
    then d[v] ← d[u] + w(u, v)
         π[v] ← u (keep track of actual path)
```

Lemma: **Assume that we initialize all $d(v)$ to ∞, $d(s) = 0$ and execute a series of Relax operations. Then for all v, $d(v) \geq \delta(v)$.**

Lemma: **Let $P = e_1, \ldots, e_k$ be a shortest path from s to v. After initialization, suppose that we relax the edges of P in order (but not necessarily consecutively). Then $d(v) = \delta(v)$.**

Example



Algorithms

Goal of an algorithm: **Relax the edges in a shortest path in order (but not necessarily consecutively).**

Algorithms

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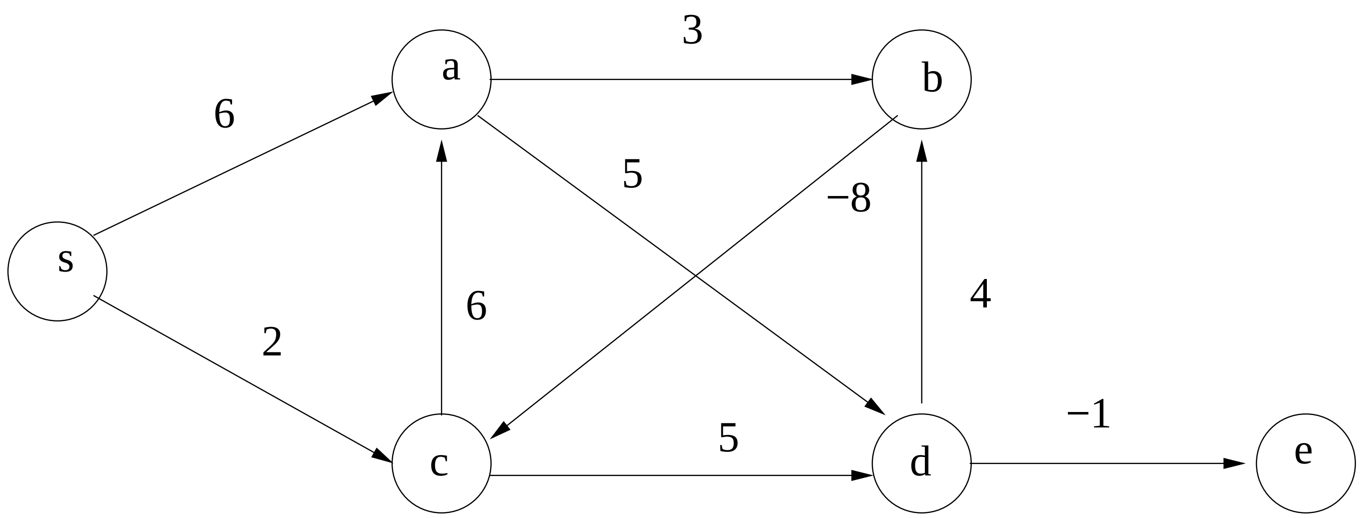
Bellman-Ford(G, w, s)

Initialize-Single-Source(G, s)
for $i \leftarrow 1$ **to** $|V[G]| - 1$
    do for each edge $(u, v) \in E[G]$
        do Relax(u, v, w)
for each edge $(u, v) \in E[G]$
    do if $d[v] > d[u] + w(u, v)$
        then return FALSE
return TRUE

$Initialize - Single - Source(G, s)$

for each vertex $v \in V[G]$
    do $d[v] \leftarrow \infty$
        $\pi[v] \leftarrow$ NIL
$d[s] \leftarrow 0$

Example



Correctness of Bellman Ford

- **Every shortest path must be relaxed in order**
- **If there are negative weight cycles, the algorithm will return false**

Running Time $O(VE)$

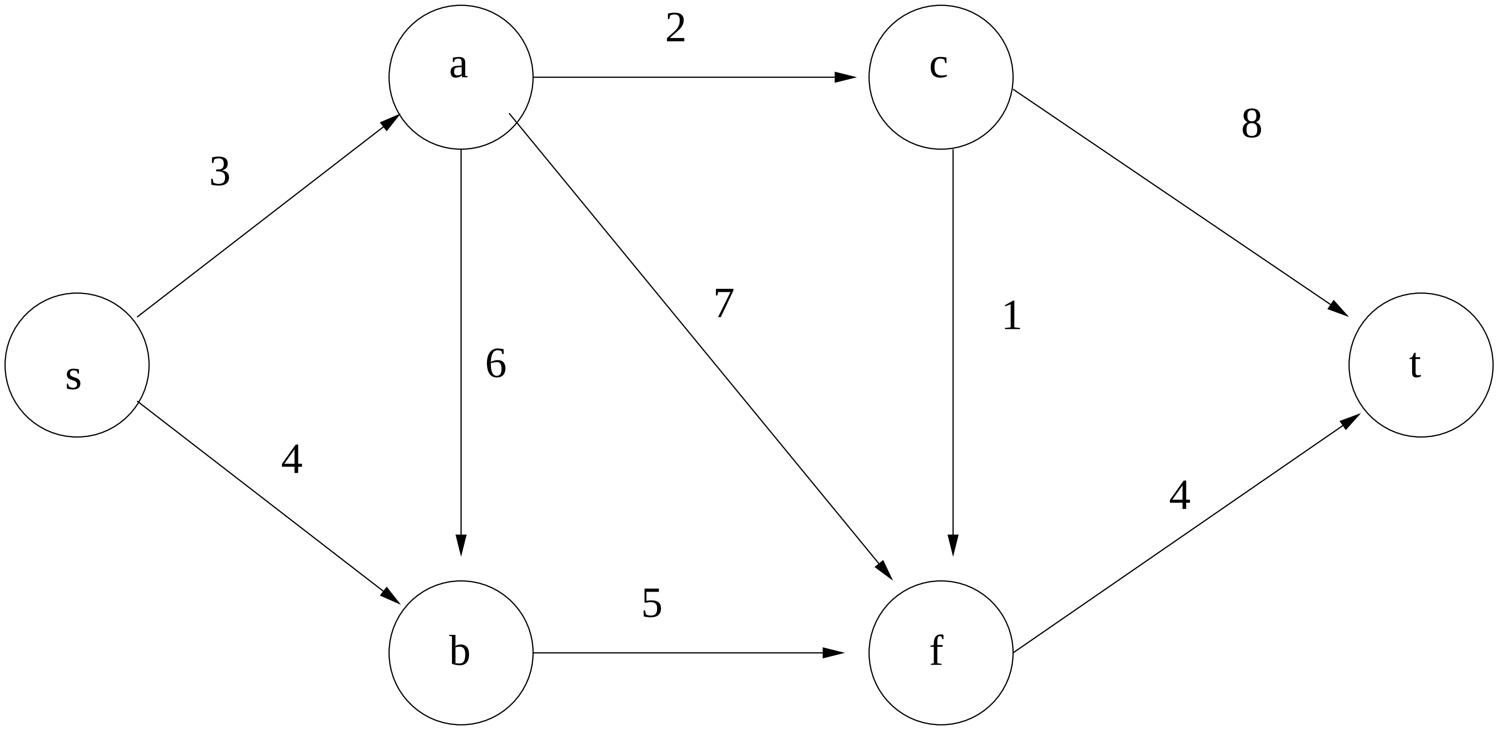
All edges non-negative

- **Dijkstra's algorithm, a greedy algorithm**
- **Similar in spirit to Prim's algorithm**
- **Idea: Run a discrete event simulation of breadth-first-search. Figure out how to implement it efficiently**
- **Can relax edges out of each vertex exactly once.**

$Dijkstra(G, w, s)$

```
Initialize-Single-Source(G, s)
S ← ∅
Q ← V[G]
while Q ≠ ∅
        do u ← Extract-Min(Q)
            S ← S ∪ {u}
            for each vertex v ∈ Adj[u]
                do Relax(u, v, w)
```

Example



Running Time and Correctness

Correctness of Dijkstra's algorithm **Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with nonnegative weight function w and source s, terminates with $d[u] = \delta(s, u)$ for all vertices $u \in V$.**

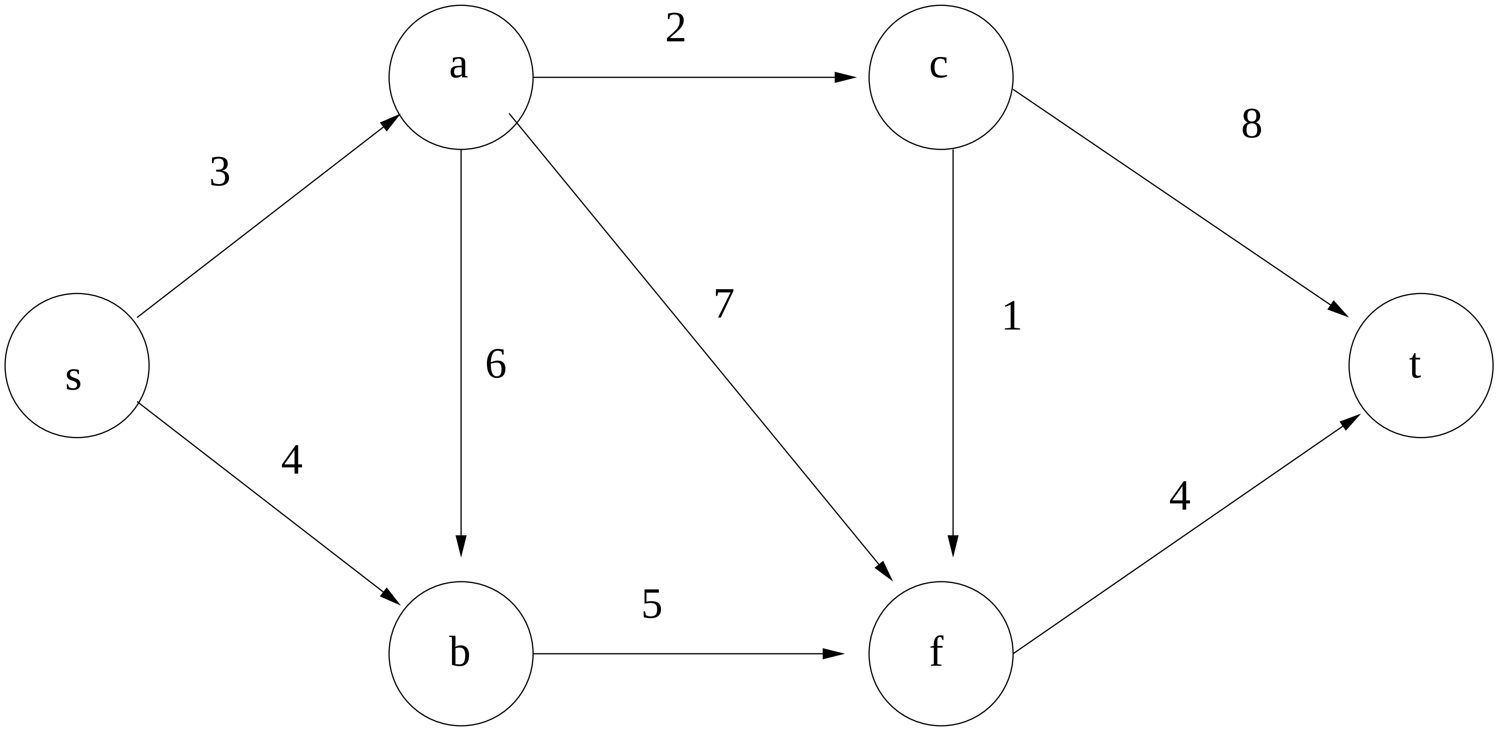
- **E decrease keys and V delete-min's**
- **$O(E \log V)$ using a heap**
- **$O(E + V \log V)$ using a Fibonacci heap**

Shortest Path in a DAG

Dag-Shortest-Paths(G, w, s)

topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE$'(G, s)$
for each u **taken in topological order**
    do for each $v \in Adj[u]$
        do RELAX(u, v, w)

Example



Correctness and Running Time

Correctness **If a weighted, directed graph $G = (V, E)$ has source vertex s and no cycles, then at the termination of the** DAG-SHORTEST-PATHS **procedure, $d[v] = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree.**

Running Time

- **Topological sort is linear time**
- **Each edge is relaxed once**
- **No additional data structure overhead**

$O(V + E)$ **time.**