Disjoint Sets

• Set of items - $X$.
• Maintain disjoint sets $S_1, \ldots, S_k$; i.e. $S_i \cap S_j = \emptyset \ \forall i \neq j$
• Operations:
  – MakeSet($x$) - create a one-element set with $x$
  – Find-Set($x$) - return the “name” of the set containing $x$
  – Union($x, y$) - merge the set containing $x$ and the set containing $y$ into one set.

Representation

• Represent set as a rooted tree, with name being root
• Time per operation is proportional to height of tree.
• Two good heuristics
  – Union by Rank - make shallow tree a child of root of big tree
  – Path Compression - every time you touch a node, make it a child of root
• Union by Rank gives $\log V$ time per operation
• Union by Rank and path compression give better performance.
Disjoint Set Code

Make-Set(x)
1    \( p[x] \leftarrow x \)
2    \( \text{rank}[x] \leftarrow 0 \)

Union(x, y)
1    \( \text{Link(Find-Set}(x), \text{Find-Set}(y)) \)

Link(x, y)
1    if \( \text{rank}[x] > \text{rank}[y] \)
2        then \( p[y] \leftarrow x \)
3    else \( p[x] \leftarrow y \)
4    if \( \text{rank}[x] = \text{rank}[y] \)
5        then \( \text{rank}[y] \leftarrow \text{rank}[y] + 1 \)

Find-Set(x)
1    if \( x \neq p[x] \)
2        then \( p[x] \leftarrow \text{Find-Set}(p[x]) \)
3    return \( p[x] \)
**Ackerman’s Function**

\[
A_k(j) = \begin{cases} 
    j + 1 & \text{if } k = 0 \\
    A_{k-1}^{(j+1)}(j) & \text{if } k \geq 1
\end{cases}
\]

\[
\alpha(n) = \min\{k : A_k(1) \geq n\}
\]

\[
A_0(j) = j + 1 \\
A_1(j) = A_0^{(j+1)}(j) = 2j + 1 \\
A_2(j) = A_1^{(j+1)}(j) = 2^{2^{2^{\cdots^{(2j + 1)\cdots}}}} + 1 + 1 = 2^{2^j + 1} - 1
\]
Ackerman

\[ A_3(1) = A_2^{(2)}(1) \]
\[ = A_2(A_2(1)) \]
\[ = A_2(7) \]
\[ = 2^8 \cdot 8 - 1 \]
\[ = 2^{11} - 1 \]
\[ = 2047 \]

\[ A_4(1) = A_3^{(2)}(1) \]
\[ = A_3(A_3(1)) \]
\[ = A_3(2047) \]
\[ = A_2^{(2048)}(2047) \]
\[ \gg A_2(2047) \]
\[ = 2^{2048} \cdot 2048 - 1 \]
\[ > 2^{2048} \]
\[ = (2^4)^{512} \]
\[ = 16^{512} \]
\[ \gg 10^{80}, \]
Summary

• Amortized time per operation is $\alpha(V)$.
• Can think of it as $\lg^* V$, which is slightly bigger.