Hiring Problem

best = 0
for i = 1 to n
    interview candidate i
    if i is better than best
        hire i
        best = i

Count # hirings. O(n) hirings
1 ≤ hirings ≤ n.

avg. is not \( \frac{n}{2} \).
Assume all orderings of candidates are equally likely.

\[ \text{expected value (}X\text{)} = \sum_{\text{outcomes of } X} \Pr(}X\text{ occurs)} \cdot \text{value (}X\text{)} . \]

\[ h = \# \text{ hirings} \]
\[ h(\pi_i) = \# \text{ hirings when the order of the candidates is permutation } \pi_i . \]

\[ E[X] = \sum_{\text{perm. } \pi_i} \frac{1}{n!} h(\pi_i) . \]

\[ = \sum_{j=1}^{n} \binom{n}{j} \Pr(\text{hire exactly } j \text{ times}) \cdot j . \]
Indicator random variables

\[ I\{A\} = \begin{cases} 1 & \text{if event occurs} \\ 0 & \text{o.w.} \end{cases} \]

Q. Expected number of heads in one coin flip.

Let \( Y \) be a r.v. denoting heads or tails.

\[ X_H = I\{Y = \text{"Heads"}\} = \begin{cases} 1 & \text{if } Y = H \\ 0 & \text{if } Y = T \end{cases} \]

\[ E[X_H] = \Pr(X_H = 1) \cdot 1 + \Pr(X_H = 0) \cdot 0 = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}. \]
\( n \) Coin Flips
\[ E[\text{# of heads}] = \sum_{i=0}^{n} \Pr(\text{exactly } i \text{ heads}) \cdot i \]

\( X_j = \# \text{ heads on flip } i \)
\( X = \sum_{j=1}^{n} X_j \)

\[ E[X] = E\left[ \sum_{j=1}^{n} X_j \right] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2}. \]
Hiring problem,

\( X_i \) = \# of candidates hired on the day when the \( i \)th is interviewed.

\( X = \) total \# of hirings

\( X = \sum_{i=1}^{n} X_i \)

\[ E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) \]

\[ E(X_i) = \Pr(X_i = 1) \cdot 1 + \Pr(X_i = 0) \cdot 0 = \Pr(\text{\( i \)th candidate is hired}) \]
\( \text{Candidate 1: } \mathbb{E}[X_1] = 1 \)

\[ \mathbb{E}[X_2] = \frac{1}{2} \]

\[ \mathbb{E}[X_3] = \frac{1}{3} \]

\[ \mathbb{E}[X_i] = \]

\[ \Pr(\text{candidate } i \text{ is better than all candidates } 1 \ldots i-1) = \frac{1}{i} \]

\[ \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n \]
Assumption: candidates come in a random order.

Remove assumption: Have the algorithm randomly order the candidates.

Bad input: increasing order

Bad case: my random permutation algorithm accidentally sorts.
Function
Random(a, b) - returns an integer between a and b, uniformly at random.
Birthday Paradox

\[ n \text{ people} \]

\[ E[\text{# of pairs w/ same birthday}] \]

\[ X_{ij} = 1 \text{ if } i \neq j \text{ having same b.d.} \]

\[ X = \# \text{ pairs w/ same b.d.} \]

\[ X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365} \]

\[ = \left( \begin{array}{c} n \\ 2 \end{array} \right) \frac{1}{365} = \frac{n(n-1)}{2} \cdot \frac{1}{365} \]

\[ n = 23 \quad 0.69 \]

\[ n = 28 \quad 1.03 \]

\[ n = 64 \quad 5.5 \]
Streaks

$n$ coins, what's the longest streak of heads?

$X_{ik} = 1$ if heads on flips $i$ to $i+k-1$

$X_k = \sum_{i=1}^{n-k+1} X_{ik} = \# \text{ of streaks of length } k$

$E[X_k] = \sum_{i=1}^{n-k+1} E[X_{ik}]$

$= \sum_{i=1}^{n-k+1} \frac{1}{2^k} = \frac{n-k+1}{2^k}$.
\[ \frac{n-k+1}{2^k} \quad \# \text{streaks of length } k \]

\[ k=3 \quad \sim \quad \frac{n}{8} \]

\[ k=n \quad \sim \quad \frac{1}{2^n} \]

\[ k = \Theta \log n \]

\[ n - \Theta \log n + 1 \]

\[ \frac{n - \Theta \log n + 1}{2^{\Theta \log n}} \quad \sim \quad \frac{n}{n^c} \quad \sim \quad \frac{1}{n^{c-1}}. \]

If \( c = \frac{1}{2} \):

\[ \sim \quad \frac{1}{n^{\frac{1}{2}}} = \sqrt{n} \]

\[ c = 4 \quad \sim \quad \frac{1}{n^4} \]

Length of longest streak = \( \Theta(\log n) \)