The Tableau Method for Simplex

The tableau method is meant to be a shorthand way of describing the operations of the Simplex method. It uses a table, or *tableau*, to display the work that is being performed. This is also the method that the text uses (as opposed to manipulating the set of equations, which is what Professor Stein has used). The general ideas of the tableau method are as follows:

- All variables will be on the left hand side, and the values of the equations on the right hand side (RHS).
- As with any Simplex method, there will be one basic variable per row. The column of any basic variable will have all zeros, except for its rows, which will have a one.
- Only the coefficients will be stored in the tableau, so it is important to keep track of which column refers to each variable. The column titles will be the variable names.

So the generic tableau will look like:

z x_1 x_2 $\dots$ x_n	RHS	Basic Variables
COEFFICIENTS	value of the RHS	Basic variable for this row
$\vdots$		

In the following, the tableau method will be shown on the example Professor Stein did in class, on 9/23.

$$\begin{array}{llllll} \text{max} & 3x_1 & +x_2 & +2x_3 & \\ \text{s.t.} & & & & \\ & x_1 & +x_2 & +3x_3 & \leq 30 \\ & 2x_1 & +2x_2 & +5x_3 & \leq 24 \\ & 4x_1 & +x_2 & +2x_3 & \leq 36 \\ & x_1, x_2, x_3 & & & \geq 0 \end{array}$$

Recall this was converted to the standard form:

$$\begin{array}{rlllllll} z & = 3x_1 & +x_2 & +2x_3 & & & & \\ & x_1 & +x_2 & +3x_3 & +s_1 & & & = 30 \\ & 2x_1 & +2x_2 & +5x_3 & & +s_2 & & = 24 \\ & 4x_1 & +x_2 & +2x_3 & & & +s_3 & = 36 \end{array}$$

Iteration 1
The corresponding tableau is:

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	−2	−1	−3	0	0	0	0	$z = 0$
0	1	1	3	1	0	0	30	$s_1 = 30$
0	2	2	5	0	1	0	24	$s_2 = 24$
0	4	1	2	0	0	1	36	$s_3 = 36$

Note that the coefficients for the non-z variables in the objective (first) row are all negative. This is the result of putting all variables on the left-hand side, i.e., this represents:

$$z - 2x_1 - x_2 - 3x_3 = 0$$

This means that the choice of improving, or entering, variables needs to be *negative* for a maximization problem (which is nonintuitive). There are three choices for entering variables, x_1, x_2 or x_3, as

each improves the objective, i.e., as it is a maximization problem, variables with negative coefficients improve the objective. x_1 will be chosen to enter.
To determine the limiting row, we use formula (10) from page 133. For all constraint equations (i.e., not the first equation), choose the minimum of:

$$\frac{\text{Right-hand side of row}}{\text{Coefficient of entering variable of row}}$$

when the *coefficient is positive*.
For this problem, the calculation is (the corresponding column will be set off with double lines):

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables	limiting row calculation
1	-2	-1	-3	0	0	0	0	$z = 0$	n/a
0	1	1	3	1	0	0	30	$s_1 = 30$	30/1
0	2	2	5	0	1	0	24	$s_2 = 24$	24/2
0	4	1	2	0	0	1	36	$s_3 = 36$	36/4

Thus, the limiting row, i.e., minimum ratio, is the fourth row, and s_3 becomes the *leaving variable*, as it is the basic variable associated with this row. Now, in order to update the tableau, row operations must be used to make the coefficients in the x_1 column zero, except for the row corresponding to s_3, which will be a one. To do this, this row will be multiplied by 1/4, resulting in the tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	-3	-1	-2	0	0	0	0	$z = 0$
0	1	1	3	1	0	0	30	$s_1 = 30$
0	2	2	5	0	1	0	24	$s_2 = 24$
0	1	1/4	1/2	0	0	1/4	9	$x_1 = 9$

and then, the rest of the x_1-column's coefficients will be eliminated via row operations. Thus, the -2 times the leaving (fourth) row will be added to the third row.

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	-3	-1	-2	0	0	0	0	$z = 0$
0	1	1	3	1	0	0	30	$s_1 = 30$
0	0	3/2	4	0	1	$-1/2$	6	$s_2 = 6$
0	1	1/4	1/2	0	0	1/4	9	$x_1 = 9$

Then, -1 times the leaving (fourth) row will be added to the second row, and -3 times the leaving row will be added to the first (objective) row.

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	0	$-1/4$	$-1/2$	0	0	3/4	27	$z = 27$
0	0	3/4	5/2	1	0	1/4	21	$s_1 = 21$
0	0	3/2	4	0	1	$-1/2$	6	$s_2 = 6$
0	1	1/4	1/2	0	0	1/4	9	$x_1 = 9$

This completes an iteration. Compare this to the corresponding set of equations at the end of the iteration:

$$\begin{aligned}
z &= 27 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_3 - \tfrac{3}{4}s_3 \\
x_1 &= 9 - \tfrac{1}{4}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{4}s_3 \\
s_1 &= 21 - \tfrac{3}{4}x_2 - \tfrac{5}{2}x_3 + \tfrac{1}{4}s_3 \\
s_2 &= 6 - \tfrac{3}{2}x_2 - 4x_3 + \tfrac{1}{2}s_3
\end{aligned}$$

Iteration 2

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	0	$-1/4$	$-1/2$	0	0	$3/4$	27	$z = 27$
0	0	$3/4$	$5/2$	1	0	$1/4$	21	$s_1 = 21$
0	0	$3/2$	4	0	1	$-1/2$	6	$s_2 = 6$
0	1	$1/4$	$1/2$	0	0	$1/4$	9	$x_1 = 9$

The variables that could improve the objective are x_2 and x_3, as they both have negative coefficients. Note that s_3 can't enter, as it would make the objective worse. As was done in the example, x_3 will be chosen, and the limiting row/leaving variable will be found:

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables	limiting row calculation
1	0	$-1/4$	$-1/2$	0	0	$3/4$	27	$z = 27$	n/a
0	0	$3/4$	$5/2$	1	0	$1/4$	21	$s_1 = 21$	$\frac{21}{\frac{5}{2}} = 82/5$
0	0	$3/2$	4	0	1	$-1/2$	6	$s_2 = 6$	$6/4 = 3/2$
0	1	$1/4$	$1/2$	0	0	$1/4$	9	$x_1 = 9$	$\frac{9}{1/2} = 18$

So, the minimum is the third row, which has s_2 as the corresponding leaving variable. As before, row operation will be used on the column associated with x_3 to make the coefficients 0 in all rows but the limiting row (third) which will be made into a 1, to result in (thus, multiply the third row by $\frac{1}{4}$, and elimate the other entries above and below):

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	0	$-1/16$	0	0	$1/8$	$11/16$	$111/4$	$z = 111/4$
0	0	$-3/16$	0	1	$-5/8$	$1/16$	$69/4$	$s_1 = 69/4$
0	0	$3/8$	1	0	$1/4$	$-1/8$	$3/2$	$x_3 = 3/2$
0	1	$1/16$	0	0	$-1/8$	$5/16$	$33/4$	$x_1 = 33/4$

The corresponding set of equations are:

$$\begin{aligned}
z &= \tfrac{111}{4} + \tfrac{1}{16}x_2 - \tfrac{1}{8}s_2 - \tfrac{11}{16}s_3 \\
x_1 &= \tfrac{33}{4} - \tfrac{1}{16}x_2 + \tfrac{1}{8}s_2 - \tfrac{5}{16}s_3 \\
x_3 &= \tfrac{3}{2} - \tfrac{3}{8}x_2 - \tfrac{1}{4}s_2 + \tfrac{1}{8}s_3 \\
s_1 &= \tfrac{69}{4} + \tfrac{3}{16}x_2 + \tfrac{5}{8}s_2 - \tfrac{1}{16}s_3
\end{aligned}$$

Iteration 3

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	0	$-1/16$	0	0	$1/8$	$11/16$	$111/4$	$z = 111/4$
0	0	$-3/16$	0	1	$-5/8$	$1/16$	$69/4$	$s_1 = 69/4$
0	0	$3/8$	1	0	$1/4$	$-1/8$	$3/2$	$x_3 = 3/2$
0	1	$1/16$	0	0	$-1/8$	$5/16$	$33/4$	$x_1 = 33/4$

In this iteration, there is only one choice for the entering variable, x_2. Computing the limiting row results in:

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables	limiting row calculation
1	0	$-1/16$	0	0	$1/8$	$11/16$	$111/4$	$z = 111/4$	n/a
0	0	$-3/16$	0	1	$-5/8$	$1/16$	$69/4$	$s_1 = 69/4$	$-3/16$ is negative!
0	0	$3/8$	1	0	$1/4$	$-1/8$	$3/2$	$x_3 = 3/2$	$\frac{3/2}{3/8} = 4$
0	1	$1/4$	$1/16$	0	$-1/8$	$5/16$	$33/4$	$x_1 = 33/4$	$\frac{33/4}{1/4} = 33$

Thus, the leaving variable is x_3. Again, row operations will be used to convert the coefficients in the column associated with x_2 to be zero, except for the the third row, which will be converted to a 1. This results in the tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Basic Variables
1	0	0	1/6	0	1/6	2/3	28	$z = 28$
0	0	0	1/2	1	1/2	0	18	$s_1 = 18$
0	0	1	8/3	0	2/3	−1/3	4	$x_2 = 4$
0	1	0	1/2	0	−1/2	0	18	$x_1 = 18$

Examining the first line, all the entries are nonnegative, which for a maximization problem, means there is no possible improvement to the objective. Hence, the current solution is optimal.

The corresponding set of equations are:

$$\begin{aligned}
z &= 28 - \tfrac{1}{6}x_3 - \tfrac{1}{6}s_2 - \tfrac{2}{3}s_3 \\
x_1 &= 8 + \tfrac{1}{6}x_3 + \tfrac{1}{6}s_2 - \tfrac{1}{3}s_3 \\
x_2 &= 4 - \tfrac{8}{3}x_3 - \tfrac{2}{3}s_2 + \tfrac{1}{3}s_3 \\
s_1 &= 18 - \tfrac{1}{2}x_3 + \tfrac{1}{2}s_2
\end{aligned}$$