

Finding feasible solutions to a LP

In all the examples we have seen until now, there was an “easy” initial basic feasible solution: put the slack variables on the left hand side. However, this is not always the case, especially for minimization problems, or problems with equality constraints in the original model.

Consider the following simple LP.

$$\begin{array}{ll} \text{minimize} & x \\ \text{s.t} & x \geq 5 \\ & x \geq 0 \end{array}$$

Forget for a minute that the solution is obvious. If we try to use simplex, we will first convert it to a max problem and negate the objective function.

$$\begin{array}{ll} \text{maximize} & -x \\ \text{s.t} & x \geq 5 \\ & x \geq 0 \end{array}$$

When we are finished, we will remember to negate the answer.

Let’s convert the constraint to an equality by adding an excess variable.

$$\begin{array}{ll} \text{maximize} & -x \\ \text{s.t} & x - e = 5 \\ & x, e \geq 0 \end{array}$$

Notice that if we try to put the excess variable on the left hand side, we get the equation

$$e = -5 + x ,$$

which would give e a value of -5 and would not yield a bfs.

To deal with this situation, we use an *artificial variable*. The artificial variable will allow us to easily find a bfs. The problem is that artificial variable may allow us to find “solutions” that are not really solutions to the original LP. To compensate, we will add the artificial variable to the objective function with a very large negative coefficient (big M). Thus, if we make the artificial variable positive, the objective function will be extremely negative. But we are maximizing, so we have every incentive to avoid making the objective function value negative, i.e. we want to avoid making a positive.

When we are done solving the LP, if the artificial variable is zero, we have solved the original LP. If it is positive, then we conclude that the original LP was infeasible.

Adding the artificial variable to the LP above, we get

$$\text{maximize} \quad -x \quad - \quad Ma \tag{1}$$

subject to

$$x - e + a = 5 \tag{2}$$

$$x, e, a \geq 0 . \tag{3}$$

In Standard form.

$$z = -x - Ma \tag{4}$$

$$a = 5 - x + e \tag{5}$$

Note that this is not in standard form. a is on the left hand side of a constraint, and the right hand side of the objective function. We can't allow this, so we use the equation to get rid of the a in the objective function. The objective function is now

$$\begin{aligned} -x - Ma &= -x - M(5 - x + e) \\ &= -5M + (M - 1)x - Me \end{aligned}$$

Rewriting, we obtain

$$z = -5M + (M - 1)x - Me \tag{6}$$

$$a = 5 - x + e \tag{7}$$

Now we have a basic feasible solution $(x, e, a) = (0, 0, 5)$ and can continue with the simplex algorithm. Remember that M is a big number. We choose x as the entering variable, and a as the leaving variable.

$$z = -5 - e + (1 - M)a \tag{8}$$

$$x = 5 + e - a \tag{9}$$

All the coefficients in the objective function are negative, so we have an optimal solution. Notice that the value of a is 0, which means that the original LP is feasible. The value of x is 5 and the objective function is -5 . Negating that we get that the optimal objective function value is 5, as we expected.

The bevco Example

Here is a more involved example, that comes from the Bevco problem in the book. In this case the LP is:

$$\text{minimize } 2x_1 + 3x_2 \tag{10}$$

subject to

$$.5x_1 + .25x_2 \leq 4 \tag{11}$$

$$x_1 + 3x_2 \geq 20 \tag{12}$$

$$x_1 + x_2 = 10 \tag{13}$$

$$x_1, x_2, x_3 \geq 0 \tag{14}$$

Note that the solution in which x_1 and x_2 are both zero (and the slacks and excesses non zero) is not feasible. We need to introduce artificial variables to help get an initial feasible solution.

We also negate the objective function and convert to a maximization problem.

$$\text{maximize } -2x_1 - 3x_2 - Ma_1 - Ma_2 \quad (15)$$

subject to

$$.5x_1 + .25x_2 + s_1 = 4 \quad (16)$$

$$x_1 + 3x_2 - e_1 + a_1 = 20 \quad (17)$$

$$x_1 + x_2 + a_2 = 10 \quad (18)$$

$$x_1, x_2, s_1, e_1, a_1, a_2 \geq 0 \quad (19)$$

Now we convert it to standard form. Either a slack variable or an artificial variable goes on the left hand side. We also will substitute for a_1 and a_2 in the objective function. This yields:

$$z = -30M + (2M - 2)x_1 + (4M - 3)x_2 - Me_1 \quad (20)$$

$$s_1 = 4 - .5x_1 - .25x_2 \quad (21)$$

$$a_1 = 20 - x_1 - 3x_2 + e_1 \quad (22)$$

$$a_2 = 10 - x_1 - x_2 \quad (23)$$

We choose x_2 as the entering variable and a_1 as the leaving variable. After an annoying amount of algebra, we obtain:

$$z = \frac{-60-10M}{3} + \frac{2M-3}{3}x_1 + \frac{M-3}{3}e_1 + \frac{3-4M}{3}a_1 \quad (24)$$

$$x_2 = \frac{20}{3} - \frac{x_1}{3} + \frac{e_1}{3} - \frac{a_1}{3} \quad (25)$$

$$s_1 = \frac{7}{3} - \frac{5}{12}x_1 - \frac{e_1}{12} + \frac{a_1}{12} \quad (26)$$

$$a_2 = \frac{10}{3} - \frac{2x_1}{3} - \frac{e_1}{3} + \frac{a_1}{3} \quad (27)$$

Now we pivot in x_1 and pivot out a_2 and obtain:

$$z = 25 - \frac{e_1}{2} + \frac{1-2M}{2}a_1 + \frac{3-2M}{2}a_2 \quad (28)$$

$$x_1 = 5 - \frac{e_1}{2} + \frac{a_1}{2} - \frac{3a_2}{2} \quad (29)$$

$$x_2 = 5 + \frac{e_1}{2} - \frac{a_1}{2} + \frac{a_2}{2} \quad (30)$$

$$s_1 = \frac{1}{4} + \frac{e_1}{8} - \frac{a_1}{8} + \frac{5a_2}{8} \quad (31)$$

Now all the coefficients in the objective row are negative, so we have an optimal solution. Also, a_1 and a_2 are both non-basic, so the problem is feasible. The optimal solution, in the original variables, is $x_1 = 5, x_2 = 5$ with objective value 25.

An infeasible LP

Let's see what happens if our original LP is infeasible. Consider the LP:

$$\text{maximize } x_1 \tag{32}$$

subject to

$$x_1 + x_2 \geq 7 \tag{33}$$

$$x_1 + x_2 \leq 6 \tag{34}$$

$$x_1, x_2 \geq 0 . \tag{35}$$

We add slack variable s_1 to the first inequality, excess and artificial variable to the second, and obtain:

$$z = x_1 - Ma_1 \tag{36}$$

$$a_1 = 7 - x_1 - x_2 + e_1 \tag{37}$$

$$s_1 = 6 - x_1 - x_2 \tag{38}$$

Next we eliminate a_1 from the objective, yielding:

$$z = -7M + (7M - 1)x_1 + 7Mx_2 - 7Me_1 \tag{39}$$

$$a_1 = 7 - x_1 - x_2 + e_1 \tag{40}$$

$$s_1 = 6 - x_1 - x_2 \tag{41}$$

Now, we choose x_2 to enter and s_1 to leave:

$$z = 35M - x_1 - 7Ms_1 - 7Me_1 \tag{42}$$

$$x_2 = 6 - x_1 - s_1 \tag{43}$$

$$a_1 = 1 + s_1 + e_1 \tag{44}$$

The simplex algorithm terminates because all the objective coefficients are negative. But, the objective function value is $35M$ and $a_1 = 1$. This tells us that the original problem is infeasible.