A Degenerate LP

An LP is *degenerate* if in a basic feasible solution, one of the basic variables takes on a zero value. Degeneracy is a problem in practice, because it makes the simplex algorithm slower.

Original LP

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 + x_3 && (1)\\ \text{subject to} \quad & && \\ & x_1 + x_2 \leq 8 && (2)\\ & -x_2 + x_3 \leq 0 && (3)\\ & x_1, x_2, \geq 0 \; . && (4) \end{aligned}$$

Standard form.

$$\begin{aligned} z &= x_1 + x_2 + x_3 && (5)\\ s_1 &= 8 - x_1 - x_2 && (6)\\ s_2 &= - x_2 + x_3 && (7) \end{aligned}$$

Note that one of the basic variables is 0. We choose x_1 as the entering variable and s_1 as the leaving variable.

$$\begin{aligned} z &= 8 + x_3 - s_1 && (8)\\ x_1 &= 8 - x_2 - s_1 && (9)\\ s_2 &= x_2 - x_3 && (10) \end{aligned}$$

Note again that one of the basic variables is 0. The previous pivot did increase the objective function value from 0 to 8 though.

We now choose x_3 as the entering variable, and s_2 as the leaving variable. These were our only choices.

$$\begin{aligned} z &= 8 + x_2 - s_1 - s_2 && (11)\\ x_1 &= 8 - x_2 - s_1 && (12)\\ x_3 &= x_2 - s_2 && (13) \end{aligned}$$

Note that the objective function did not increase. This occurs because of degeneracy.

We now choose x_2 as the entering variable and x_1 as the leaving variable.

$$\begin{aligned} z &= 16 - x_1 - 2s_1 - s_2 && (14)\\ x_2 &= 8 - x_1 - s_1 && (15)\\ x_3 &= 8 - x_1 - s_1 - s_2 && (16) \end{aligned}$$

Since all coefficients of variables in the objective function are negative, we now have the optimal solution, $(x_1, x_2, x_3, s_1, s_2) = (0, 8, 8, 0, 0)$ with objective value 16. Notice that in the final solution, the basic variables are all non-zero. In a degenerate LP, it is also possible that even in the final solution, some of the basic variables will be zero.

One other thing to note is that x_1 was an entering variable in one iteration, and a leaving variable in another. In general, a variable can be an entering and leave the basic many times in the course of the simplex algorithm.