Survivor Game

Let $f(x)$ be the outcome, if you start at x.
Let $m(x)$ be the optimal, move if you start at x.

number left	$f(x)$	$m(x)$
1	W	1
2	W	2
3	W	3
4	L	anything
5	W	1
6	W	2
7	W	3
8	L	anything
9	W	1
10	W	2
11	W	3
12	L	anything
13	W	1
14	W	2
15	W	3
16	L	anything
17	W	1
18	W	2
19	W	3
20	L	anything
21	W	1

We can develop a recurrence for the win/loss function.

$$f(x) = \begin{cases} W & \text{if } x = 1, 2, 3 \\ L & \text{if } x = 4 \\ f(x-4) & \text{otherwise} \end{cases}$$

$m(x)$ will be the move that takes you to the place described by the f function.

Scheduling Problem

Producing DVD's of the TV show survivor is a four step process. Probstco has two assembly lines that perform the process. The times are listed below. (We use $c_t(x)$ to denote the processing time of line x at step t.

Step	Time on Line 1	Time on Line 2
1	60	65
2	65	45
3	45	40
4	30	35

A DVD can switch between the two assembly lines, but it takes 10 minutes to switch between lines. It takes no time to progress along the same line.

Figure out a sequence which minimizes the total time.

Solution

It's not clear how to work from the beginning, but it's easier to think about working backwards from the end.

Let $f_t(x)$ be the fastest time from step t to the end, starting on line x.
$f_4(1) = 30$
$f_4(2) = 35$

How do we compute the other values? We use the already computed values of f for larger values of t.

Think about $f_3(1)$. There are two possibilities for the fastest time to the end. You either stay on machine 1 next, or switch to machine 2. If you stay on 1, the total time is $45 + 30 = 75$, if you switch, the time is $45 + 10 + 35 = 85$. So you would choose to stay on line 1.

In symbols

$$f_3(1) = \min\{c_3(1) + f_4(1), c_3(1) + 10 + f_4(2)\}.$$

Symmetrically,

$$f_3(2) = \min\{c_3(2) + f_4(2), c_3(2) + 10 + f_4(1)\}.$$

In general,

$$f_t(x) = \min\{c_t(x) + f_{t+1}(x), c_t(x) + 10 + f_{t+1}(3 - x)\}.$$

We can fill in a table, using this formula.

Step	$f_t(1)$	$f_t(2)$
1	190	185
2	140	120
3	75	75
4	30	35

So the shortest time starting on line 1 is 190, and the shortest starting on line 2 is 185. One can trace back and figure out how the decisions where made.

Dynamic Programming

Dynamic Programming is a technique for solving problems with certain features. These are:

- The problem has a series of stages, each with an associated decision.
- Each stage has a number of states associated with it.
- The decision at each stage descirbes which state to choose next.
- Given a particular state, future decisions do not depend on anything except that state. They do not depend on how the state was reached, or previous states along the path. (optimal substructure)
- The number of states and stages is limited.

Inventory Problem

A state can be defined by the inventory and the period.

Let $f_t(i)$ be the minimum cost of meeting demands for months $t, t+1, \ldots, 4$ starting with inventory i at the beginning of month t.

We work backwards, starting with period 4. The demand for the this period is 4.

If we have zero inventory, we have to pay a fixed cost of 3 plus the cost of producing 4 units.

$$f_4(0) = 3 + 4 = 7$$

For larger inventories we have smaller costs:

$$\begin{aligned} f_4(0) &= 3 + 4 = 7 \\ f_4(1) &= 3 + 3 = 6 \\ f_4(2) &= 3 + 2 = 5 \\ f_4(3) &= 3 + 1 = 4 \\ f_4(4) &= 0 \end{aligned}$$

Note that for an inventory of 4 we don't have any costs. This makes the problem non-linear.

We can start to fill in a table:

i/t	1	2	3	4
0				7
1				6
2				5
3				4
4				0

Now let's think about $f_3(0)$. We must satisfy the demand of 2, but we now have choices, because we can end up with different inventory amounts. How do we evaluate these different inventory amounts? We use the entries $f_4()$, which tell us the cost, in period 4 starting with a particular inventory amount.

We make a table:

left over	0	1	2	3	4
number produced	2	3	4	5	impossible
production cost	5	6	7	8	
inventory cost	0	.5	1	1.5	
f_4(inventory)	7	6	5	4	
total cost	12	12.5	13	13.5	

Thus the best strategy is to produce 2 and end the period with no inventory.

Now let's think about $f_3(1)$. We must satisfy the demand of 2, but we now have choices, because we can end up with different inventory amounts. How do we evaluate these different inventory amounts? We use the entries $f_4()$, which tell us the cost, in period 4 starting with a particular inventory amount.

We make a table:

left over	0	1	2	3	4
number produced	1	2	3	4	5
production cost	4	5	6	7	8
inventory cost	0	.5	1	1.5	2
f_4(inventory)	7	6	5	4	0
total cost	11	11.5	12	12.5	10

Notice that now the optimal policy is to produce 5 and end the period with an inventory of 4.

Consider now $f_3(2)$. Let's make a similar table.

left over	0	1	2	3	4
number produced	0	1	2	3	4
production cost	0	4	5	6	7
inventory cost	0	.5	1	1.5	2
f_4(inventory)	7	6	5	4	0
total cost	7	10.5	11	11.5	9

Notice that now the optimal policy is to produce 0 and end the period with an inventory of 0.

We can compute $f_3(3)$ and $f_3(4)$ similarly and fill in the master table more.

i/t	1	2	3	4
0			12	7
1			10	6
2			7	5
3			6.5	4
4			6	0

We now continue similarly for $f_2()$. We'll do $f_2(0)$.

left over	0	1	2	3	4
number produced	3	4	5	impossible	impossible
production cost	7	8	9		
inventory cost	0	.5	1		
f_3(inventory)	12	10	7		
total cost	18	17.5	16		

So the optimal policy is to manufacture 5 and end with an inventory of 2.

Notice that to compute $f_2()$, we used $f_3()$ **but not** $f_4()$. This is why DP is so efficient!

In general, we have a recurrence:

$$f_t(t) = \min_x \left\{ \frac{1}{2}(i + x - \text{demand}(t)) + \text{prod-cost}(x) + f_{t+1}(i + x - \text{demand}(t)) \right\}$$

With appropriate boundary conditions. We can continue and fill in the entire master table.

i/t	1	2	3	4
0	**20**	**16**	12	**7**
1	16	15	10	6
2	15.5	14	**7**	5
3	15	12	6.5	4
4	13.5	10.5	6	0

The bold entry represent the optimal choices. These correspond to producing $1, 5, 0, 4$.