A Degenerate LP

**Definition:** An LP is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value. Degeneracy is a problem in practice, because it makes the simplex algorithm slower.

**Original LP**

\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \\
& x_1 + x_2 \leq 8 \\
& -x_2 + x_3 \leq 0 \\
& x_1, x_2, \geq 0.
\end{align*}
\]

**Standard form.**

\[
\begin{align*}
z &= x_1 + x_2 + x_3 \\
s_1 &= 8 - x_1 - x_2 \\
s_2 &= -x_2 + x_3
\end{align*}
\]
**Iteration 1**

\[ z = x_1 + x_2 + x_3 \]  \hspace{1cm} (8)

\[ s_1 = 8 - x_1 - x_2 \]  \hspace{1cm} (9)

\[ s_2 = -x_2 + x_3 \]  \hspace{1cm} (10)

Note that one of the basic variables is 0. We choose \( x_1 \) as the entering variable and \( s_1 \) as the leaving variable.

\[ z = 8 + x_3 - s_1 \]  \hspace{1cm} (11)

\[ x_1 = 8 - x_2 - s_1 \]  \hspace{1cm} (12)

\[ s_2 = x_2 - x_3 \]  \hspace{1cm} (13)

Note again that one of the basic variables is 0. The previous pivot did increase the objective function value from 0 to 8 though.
Iteration 2

\[
\begin{align*}
    z &= 8 + x_3 - s_1 \\
    x_1 &= 8 - x_2 - s_1 \\
    s_2 &= x_2 - x_3
\end{align*}
\] (14) (15) (16)

We now choose \( x_3 \) as the entering variable, and \( s_2 \) as the leaving variable. These were our only choices.

\[
\begin{align*}
    z &= 8 + x_2 - s_1 - s_2 \\
    x_1 &= 8 - x_2 - s_1 \\
    x_3 &= x_2 - s_2
\end{align*}
\] (17) (18) (19)

Note that the objective function did not increase. This occurs because of degeneracy.
Iteration 3

\[ z = 8 + x_2 - s_1 - s_2 \]  \hspace{1cm} (20)
\[ x_1 = 8 - x_2 - s_1 \]  \hspace{1cm} (21)
\[ x_3 = x_2 - s_2 \]  \hspace{1cm} (22)

We now choose \( x_2 \) as the entering variable and \( x_1 \) as the leaving variable.

\[ z = 16 - x_1 - 2s_1 - s_2 \]  \hspace{1cm} (23)
\[ x_2 = 8 - x_1 - s_1 \]  \hspace{1cm} (24)
\[ x_3 = 8 - x_1 - s_1 - s_2 \]  \hspace{1cm} (25)

Since all coefficients of variables in the objective function are negative, we now have the optimal solution, \((x_1, x_2, x_3, s_1, s_2) = (0, 8, 8, 0, 0)\) with objective value 16. Notice that in the final solution, the basic variables are all non-zero. In a degenerate LP, it is also possible that even in the final solution, some of the basic variables will be zero.

One other thing to note is that \( x_1 \) was an entering variable in one iteration, and a leaving variable in another. In general, a variable can be an entering and leave the basic many times in the course of the simplex algorithm.