Dual Simplex

Suppose we have the Pinocchio LP:

$$
\begin{array}{llllll}
\textbf{maximize} & 3x_1 & + & 2x_2 & & \\
\textbf{s.t.} & 2x_1 & + & x_2 & \leq & 100 \\
 & x_1 & + & x_2 & \leq & 80 \\
 & x_1 & & & \leq & 40
\end{array}
\begin{array}{r}
(1)\\ (2)\\ (3)\\ (4)
\end{array}
$$

Let's take the dual (and introduce big M):

$$
\begin{array}{llllllllllll}
\textbf{minimize} & 100y_1 & + & 80y_2 & + & 40y_3 & + & Ma_1 & + & Ma_2 & & \\
 & 2y_1 & + & y_2 & + & y_3 & - & e_1 & + & a_1 & \geq & 3 \\
 & y_1 & + & y_2 & & & - & e_2 & + & a_2 & \geq & 2
\end{array}
\begin{array}{r}
(5)\\ (6)\\ (7)
\end{array}
$$

We could negate the objective, and follow through with the big-M's, but let's see if we can work directly on the dual.

Dual Simplex

Let's remove the artificial variable and make the excess variables be the basic variables:

$$
\begin{aligned}
\textbf{min} \quad z &= \phantom{-3 +{}} 100y_1 + 80y_2 + 40y_3 && (8)\\
e_1 &= -3 + 2y_1 + y_2 + y_3 && (9)\\
e_2 &= -2 + y_1 + y_2 && (10)\\
& \quad (11)
\end{aligned}
$$

Note that the basic variables are negative. Also the objective function is too small. Let's do simplex in a different way. We will choose a negative basic variable, and try to increase it, while keeping all the objective function coefficients positive. So we choose e_1 to leave, and we choose the variable with the minimum ratio of objective function coefficient to row of e_1 coefficient (among the variables with positive e_1 row coefficient) to enter. This means that y_3, with ratio 40 will enter. We perform the pivot and obtain:

$$
\begin{aligned}
\textbf{min} \quad z &= 120 + 20y_1 + 40y_2 + 40e_1 && (12)\\
y_3 &= 3 - 2y_1 - y_2 + e_1 && (13)\\
e_2 &= -2 + y_1 + y_2 && (14)
\end{aligned}
$$

Dual Simplex, continued

$$
\begin{array}{lrcrcrcrcr}
\textbf{min} & z & = & 120 & + & 20y_1 & + & 40y_2 & + & 40e_1 \\
& y_3 & = & 3 & - & 2y_1 & - & y_2 & + & e_1 \\
& e_2 & = & -2 & + & y_1 & + & y_2 & &
\end{array}
\quad (15)\text{–}(17)
$$

Notice that the coefficient of e_1 is 40 and the objective function is 120, corresponding to the primal solution $(40, 0)$. We continue pivoting. e_2 is negative and will leave. To enter, we choose y_1 with ratio 20.

This yields

$$
\begin{array}{lrcrcrcrcr}
\textbf{min} & z & = & 160 & + & 20y_2 & + & 40e_1 & + & 20e_2 \\
& y_1 & = & 2 & - & y_2 & + & & + & e_2 \\
& y_3 & = & -1 & + & y_2 & + & e_1 & - & 2e_2
\end{array}
\quad (18)\text{–}(20)
$$

Dual Simplex

$$\begin{array}{lrcrcrcrcr} \textbf{min} & z & = & 160 & + & 20y_2 & + & 40e_1 & + & 20e_2 \\ & y_1 & = & 2 & - & y_2 & + & & + & e_2 \\ & y_3 & = & -1 & + & y_2 & + & e_1 & - & 2e_2 \end{array} \quad \begin{array}{r} (21) \\ (22) \\ (23) \end{array}$$

We perform another iteration, with y_3 leaving and y_2 entering. This yields:

$$\begin{array}{lrcrcrcrcr} \textbf{min} & z & = & 180 & + & 20y_3 & + & 20e_1 & + & 60e_2 \\ & y_1 & = & 1 & - & y_3 & + & e_1 & - & e_2 \\ & y_2 & = & 1 & + & y_3 & - & e_1 & + & 2e_2 \end{array} \quad \begin{array}{r} (24) \\ (25) \\ (26) \end{array}$$

Notice now that all the basic variables are positive. We can now stop with a dual feasible and optimal solution (1,1). Note that the objective is 180, and that the primal solution (20,60) is the coefficients of e_1 and e_2.

Dual Simplex Summary

We have just executed dual simplex, which maintains an infeasible solution, while keeping the objective function coefficients positive. What is really going on is we are maintaining a dual feasible solution (in this case the original Pinocchio primal). See the book for the details of the method.

Why use dual simplex?

- Adding a new constraint to a solved LP.
- Finding a new solution after the right hand side changes.
- Solving min problems without bigM
- For efficiency. The number of iterations tends to be proportional to the number of constraints. So if you have lots of constraints and few variables, use dual simplex.