Finding the optimal basis

Example

$$
\begin{aligned}
z = \quad & x_1 + 4x_2 \\
& x_1 + 2x_2 + s_1 &= 6 \\
& 2x_1 + x_2 + s_2 &= 8 \\
& x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}
$$

Suppose that in the optimal basis, the basic variable are x_2 and s_2. Then,

$$
x_{BV} = \begin{pmatrix} x_2 \\ s_2 \end{pmatrix} \quad x_{NBV} = \begin{pmatrix} x_1 \\ s_1 \end{pmatrix}
$$

$$
c_{BV} = (4\ 0) \quad x_{NBV} = (1\ 0)
$$

$$
B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 8 \end{pmatrix}.
$$

We compute

$$
B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}.
$$

•

The optimal final tableaux can be computed from the formulas:

$$
\begin{aligned}
z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\
x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b
\end{aligned}
$$

Computing the optimal basis

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

We compute

$$B^{-1}N = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}.$$

$$B^{-1}b = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$c_{BV}(B^{-1}N) = (4\ 0) \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} = (2\ 2)$$

$$c_{BV}(B^{-1}b) = (4\ 0) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 12$$

We now have the objective function

$$z + ((2\ 2) - (1\ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

and the constraints

$$\begin{pmatrix} x_2 \\ s_2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Final tableax

We now have the objective function

$$z + ((2\ 2) - (1\ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

and the constraints

$$\begin{pmatrix} x_2 \\ s_2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

which yields

$$\begin{aligned} z + x_1 + 2s_1 &= 12 \\ (1/2)x_1 + x_2 + (1/2)s_1 &= 3 \\ (3/2)x_1 - (1/2)s_1 + s_2 &= 5 \end{aligned}$$

Changes in parameters

- Let's explore changes in right hand side and objective function coefficient.
- We will again ask when current basis remains optimal.
  - non-negative r.h.s.,
  - non-negative coefficient in row 0
- We proceed by example

Change in objective function coeficient of a nbv

Let's change the objective function coefficient of x_1. x_1 is non-basic, so we are changing only c_{NBV}.

It become $(1 + \Delta\ 0)$. Looking at equations:

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

we see that c_{NBV} only changes objective function.

We recompute objective function. Instead of

$$z + ((2\ 2) - (1\ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

we have

$$z + ((2\ 2) - (1 + \Delta\ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

or

$$z + (1 - \Delta)x_1 + 2s_1 = 12$$

.

So, to maintain current basis, we have $\Delta \leq 1$. Note that since x_1 is nonbasic, this is reduced cost, the amount we have to change the objective function in order to make the variable basic.

Changing the coefficent of a basic variable

Let's change the coefficient of x_2

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

Again, we only have a change in objective function. A change to c_{BV} affects the objective function in two places, so we substitute $4+\Delta$ for 4 as the first entry in c_{BV} and obtain:

$$z + \left((4+\Delta\ 0)\begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} - (1\ 0)\right)\begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = (4+\Delta\ 0)\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

or

$$z + (1+\Delta/2)x_1 + (2+\Delta/2)s_1 = 12 + 3\Delta$$

So, to keep the objective function coefficients non-negative, we must have $\Delta \geq -2$ and that each increase of the coefficent by 1 increases the objective function by 3.

Right hand side change

A change to the right hand side changes b. Let's try to change 6 to $6+\Delta$.

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

Note that b only appears on the right hand side.

Changing b to $\begin{pmatrix} 6+\Delta \\ 8 \end{pmatrix}$ changes $B^{-1}b$ to $\begin{pmatrix} 3+\Delta/2 \\ 5-\Delta/2 \end{pmatrix}$. Thus the system of equations becomes

$$\begin{aligned} (1/2)x_1 + x_2 + (1/2)s_1 &= 3+\Delta/2 \\ (3/2)x_1 - (1/2)s_1 + s_2 &= 5-\Delta/2 \end{aligned}$$

and the objective function value is now

$$c_{BV}B^{-1}b = (4\ 0)\begin{pmatrix} 3+\Delta/2 \\ 5-\Delta/2 \end{pmatrix} = 12 + 2\Delta$$

So we can change Δ in the range $-6 \le \Delta \le 10$ and the shadow price is 2.

Right hand side change

What about changing the 8 to $8+\Delta$. Now

$$B^{-1}b = \begin{pmatrix} 3 \\ 5+\Delta \end{pmatrix}.$$

Our allowed range of change is $-5 \leq \Delta \leq \infty$.
What about the effect on the objective function?
We compute

$$c_{BV}B^{-1}b = (4\ 0)\begin{pmatrix} 3 \\ 5+\Delta \end{pmatrix} = 12$$

So changing the right hand side of the second equation has no effect. This is not surprising, since s_2 is basic, meaning that there is slack in the second equation.