Finding the optimal basis

Example

\[ z = x_1 + 4x_2 \]
\[ x_1 + 2x_2 + s_1 = 6 \]
\[ 2x_1 + x_2 + s_2 = 8 \]
\[ x_1, x_2, s_1, s_2 \geq 0 \]

Suppose that in the optimal basis, the basic variable are \( x_2 \) and \( s_2 \). Then,

\[ x_{BV} = \begin{pmatrix} x_2 \\ s_2 \end{pmatrix} \quad x_{NBV} = \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} \]
\[ c_{BV} = (4 \ 0) \quad c_{NBV} = (1 \ 0) \]

\[ B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \]

We compute

\[ B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}. \]

The optimal final tableaux can be computed from the formulas:

\[ z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} = c_{BV}B^{-1}b \]
\[ x_{BV} + B^{-1}Nx_{NBV} = B^{-1}b \]
Computing the optimal basis

\[ z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} = c_{BV}B^{-1}b \]

\[ x_{BV} + B^{-1}Nx_{NBV} = B^{-1}b \]

We compute

\[ B^{-1}N = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}. \]

\[ B^{-1}b = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \]

\[ c_{BV}(B^{-1}N) = (4, 0) \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} = (2, 2) \]

\[ c_{BV}(B^{-1}b) = (4, 0) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 12 \]

We now have the objective function

\[ z + ((2, 2) - (1, 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12 \]

and the constraints

\[ \begin{pmatrix} x_2 \\ s_2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \]
We now have the objective function

\[ z + ((2 \ 2) - (1 \ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12 \]

and the constraints

\[ \begin{pmatrix} x_2 \\ s_2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \]

which yields

\[ z + x_1 + 2s_1 = 12 \]
\[ (1/2)x_1 + x_2 + (1/2)s_1 = 3 \]
\[ (3/2)x_1 - (1/2)s_1 + s_2 = 5 \]
Changes in parameters

- Let’s explore changes in right hand side and objective function coefficient.
- We will again ask when current basis remains optimal.
  - non-negative r.h.s.,
  - non-negative coefficient in row 0
- We proceed by example
Let’s change the objective function coefficient of $x_1$. $x_1$ is non-basic, so we are changing only $c_{NBV}$.

It becomes $(1 + \Delta 0)$. Looking at equations:

\[
\begin{align*}
z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\
x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b
\end{align*}
\]

we see that $c_{NBV}$ only changes the objective function.

We recompute the objective function. Instead of

\[
\begin{align*}
z + ((2 \ 2) - (1 \ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} &= 12
\end{align*}
\]

we have

\[
\begin{align*}
z + ((2 \ 2) - (1 + \Delta \ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} &= 12
\end{align*}
\]

or

\[
\begin{align*}
z + (1 - \Delta)x_1 + 2s_1 &= 12
\end{align*}
\]

So, to maintain the current basis, we have $\Delta \leq 1$. Note that since $x_1$ is nonbasic, this is the reduced cost, the amount we have to change the objective function in order to make the variable basic.
Let’s change the coefficient of \( x_2 \)

\[
\begin{align*}
z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\
x_{BV} + B^{-1}N x_{NBV} &= B^{-1}b
\end{align*}
\]

Again, we only have a change in objective function. A change to \( c_{BV} \) affects the objective function in two places, so we substitute \( 4 + \Delta \) for 4 as the first entry in \( c_{BV} \) and obtain:

\[
z + ((4 + \Delta) 0) \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} - (1 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = (4 + \Delta 0) \begin{pmatrix} 3 \\ 5 \end{pmatrix}
\]

or

\[
z + (1 + \Delta/2)x_1 + (2 + \Delta/2)s_1 = 12 + 3\Delta
\]

So, to keep the objective function coefficients non-negative, we must have \( \Delta \geq -2 \) and that each increase of the coefficient by 1 increases the objective function by 3.
Right hand side change

A change to the right hand side changes $b$. Let’s try to change 6 to $6 + \Delta$.

$$z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} = c_{BV}B^{-1}b$$
$$x_{BV} + B^{-1}Nx_{NBV} = B^{-1}b$$

Note that $b$ only appears on the right hand side.

Changing $b$ to \(\begin{pmatrix} 6 + \Delta \\ 8 \end{pmatrix}\) changes $B^{-1}b$ to \(\begin{pmatrix} 3 + \Delta/2 \\ 5 - \Delta/2 \end{pmatrix}\). Thus the system of equations becomes

\[
\begin{align*}
(1/2)x_1 + x_2 + (1/2)s_1 & = 3 + \Delta/2 \\
(3/2)x_1 - (1/2)s_1 + s_2 & = 5 - \Delta/2
\end{align*}
\]

and the objective function value is now

$$c_{BV}B^{-1}b = \begin{pmatrix} 4 \\ 0 \end{pmatrix}\left(\begin{array}{c}
3 + \Delta/2 \\
5 - \Delta/2
\end{array}\right) = 12 + 2\Delta$$

So we can change $\Delta$ in the range $-6 \leq \Delta \leq 10$ and the shadow price is 2.
Right hand side change

What about changing the 8 to $8 + \Delta$. Now

$$B^{-1}b = \begin{pmatrix} 3 \\ 5 + \Delta \end{pmatrix}.$$

Our allowed range of change is $-5 \leq \Delta \leq \infty$.

What about the effect on the objective function?

We compute

$$c_{BV}B^{-1}b = (4 \ 0) \begin{pmatrix} 3 \\ 5 + \Delta \end{pmatrix} = 12$$

So changing the right hand side of the second equation has no effect. This is not surprising, since $s_2$ is basic, meaning that there is slack in the second equation.