A Degenerate LP

Definition: **An LP is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value. Degeneracy is a problem in practice, because it makes the simplex algorithm slower.**

Original LP

$$
\begin{array}{lrcl}
\textbf{maximize} & x_1 + x_2 + x_3 & & \qquad (1)\\
\textbf{subject to} & & & \\
 & x_1 + x_2 & \leq 8 & \qquad (2)\\
 & -x_2 + x_3 & \leq 0 & \qquad (3)\\
 & x_1, x_2, & \geq 0\,. & \qquad (4)
\end{array}
$$

Standard form.

$$
\begin{aligned}
z &= x_1 + x_2 + x_3 && (5)\\
s_1 &= 8 - x_1 - x_2 && (6)\\
s_2 &= - x_2 + x_3 && (7)
\end{aligned}
$$

Iteration 1

$$
\begin{aligned}
z &= \phantom{8 -{}} x_1 + x_2 + x_3 && (8)\\
s_1 &= 8 - x_1 - x_2 && (9)\\
s_2 &= - x_2 + x_3 && (10)
\end{aligned}
$$

Note that one of the basic variables is 0. We choose x_1 as the entering variable and s_1 as the leaving variable.

$$
\begin{aligned}
z &= 8 + x_3 - s_1 && (11)\\
x_1 &= 8 - x_2 - s_1 && (12)\\
s_2 &= \phantom{8 -{}} x_2 - x_3 && (13)
\end{aligned}
$$

Note again that one of the basic variables is 0. The previous pivot did increase the objective function value from 0 to 8 though.

Iteration 2

$$
\begin{aligned}
z &= 8 && && + x_3 && - s_1 && (14)\\
x_1 &= 8 && - x_2 && && - s_1 && (15)\\
s_2 &= && x_2 && - x_3 && && (16)
\end{aligned}
$$

We now choose x_3 as the entering variable, and s_2 as the leaving variable. These were our only choices.

$$
\begin{aligned}
z &= 8 && + x_2 && - s_1 && - s_2 && (17)\\
x_1 &= 8 && - x_2 && - s_1 && && (18)\\
x_3 &= && x_2 && && - s_2 && (19)
\end{aligned}
$$

Note that the objective function did not increase. This occurs because of degeneracy.

Iteration 3

$$
\begin{aligned}
z &= 8 + x_2 - s_1 - s_2 && (20)\\
x_1 &= 8 - x_2 - s_1 && (21)\\
x_3 &= x_2 - s_2 && (22)
\end{aligned}
$$

We now choose x_2 as the entering variable and x_1 as the leaving variable.

$$
\begin{aligned}
z &= 16 - x_1 - 2s_1 - s_2 && (23)\\
x_2 &= 8 - x_1 - s_1 && (24)\\
x_3 &= 8 - x_1 - s_1 - s_2 && (25)
\end{aligned}
$$

Since all coefficients of variables in the objective function are negative, we now have the optimal solution, $(x_1, x_2, x_3, s_1, s_2) = (0, 8, 8, 0, 0)$ with objective value 16. Notice that in the final solution, the basic variables are all non-zero. In a degenerate LP, it is also possible that even in the final solution, some of the basic variables will be zero.

One other thing to note is that x_1 was an entering variable in one iteration, and a leaving variable in another. In general, a variable can be an entering and leave the basic many times in the course of the simplex algorithm.