Definition of a Linear Program

Definition: **A function $f(x_1, x_2, \ldots, x_n)$ of $x_1, x_2, \ldots, x_n$ is a linear function if and only if for some set of constants $c_1, c_2, \ldots, c_n$,**

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \ .$$

Examples:

- x_1
- $5x_1 + 6x_4 - 2x_2 + 1$
- 3

Non-examples:

- x_1^2
- $x_1 + 3x_2 - 4x_3^4$
- x_1x_2

Linear Inequalities

Definition: For any linear function $f(x_1, x_2, \ldots, x_n)$ and any number b, the inequalities

$$f(x_1, x_2, \ldots, x_n) \leq b$$

and

$$f(x_1, x_2, \ldots, x_n) \geq b$$

are linear inequalities.

Examples:

- $x_1 + x_2 \leq 4$
- $5x_1 - 4 \geq 0$

Note: If an inequality can be rewritten as a linear inequality then it is one. Thus $x_1 + x_2 \leq 3x_3$ is a linear inequality because it can be rewritten as $x_1 + x_2 - 3x_3 \leq 0$. Even $x_1/x_2 \leq 4$ is a linear inequality because it can be rewritten as $x_1 - 4x_2 \leq 0$. Note that $x_1/x_2 + x_3 \leq 4$ is not a linear inequality, however.

Definition: For any linear function $f(x_1, x_2, \ldots, x_n)$ and any number b, the equality

$$f(x_1, x_2, \ldots, x_n) = b$$

is a linear equality.

LPs

Definition: A linear programming problem (LP) is an optimization problem for which:

1. We attempt to maximize (or minimize) a linear function of the decision variables. (objective function)
2. The values of the decision variables must satisfy a set of constraints, each of which must be a linear inequality or linear equality.
3. A sign restriction on each variable. For each variable x_i the sign restriction can either say
   (a) $x_i \geq 0$,
   (b) $x_i \leq 0$,
   (c) x_i unrestricted (urs).

Definition: A solution to a linear program is a setting of the variables.

Definition: A feasible solution to a linear program is a solution that satisfies all constraints.

Definition: The feasible region in a linear program is the set of all possible feasible solutions.

Definition: An optimal solution to a linear program is the feasible solution with the largest objective function value (for a maximization problem).

Modeling Assumptions for Linear Programming

- **Prportionality.** If one item brings in a profit of x, then k items bring in a profit of kx. If one item use y units of resource R then k items use ky units of resource R.
- **Additivity.** The decisions made are independent, except as noted in the constraints. So, if we sell more trains, this does not decrease the demand for soldiers, in the Giapeto model.
- **Divisibility.** Decision variables can take on fractional values.
- **Certainty.** The values of various parameters are known with certainty.

Comments:

- Whether these assumptions hold is a feature of the model, not of linear programming itself.
- They often do not hold.
- They may be close to holding, or may hold in the region we are about: e.g.
  - proportionally and additivity may hold in the feasible region
  - divisibility may not hold, but the conclusions of the model will be approximately sound anyway

– certainty may not hold, but we may have good estimates

Whenever we solve a model using linear programming, we should be aware of these assumptions, and ask ourselves whether they hold, and whether the solution makes sense.

Geometry Definitions

Definition: A set of points S is a convex set if the line segment joining any 2 points in S is wholly contained in S.

Fact: The set of feasible solutions to an LP (feasible region) forms a (possibly unbounded) convex set.

Definition: A point p of a contex set S is an extreme point if each line segment that lies completely in S and contains p has p as an endpoint. An extreme point is also called a corner point.

Fact: Every linear program has an extreme point that is an optimal solution.

Corrolary: An algorithm to solve a linear program only needs to consider extreme points.

Definition: A constraint of a linear program is binding at a point p if the inequality is met with equality at p. It is nonbinding otherwise. (Recall that a point is the same as a solution.)