Finding the optimal basis

Consider the LP:

$$
\begin{array}{llllllll}
\textbf{maximize} & 60x_1 & + & 30x_2 & + & 20x_3 & & \qquad (1) \\
\textbf{subject to} & & & & & & & \\
& 8x_1 & + & 6x_2 & + & x_3 & + s_1 \qquad\qquad = 48 & \qquad (2) \\
& 4x_1 & + & 2x_2 & + & 1.5x_3 & \qquad + s_2 \qquad = 20 & \qquad (3) \\
& 2x_1 & + & 1.5x_2 & + & .5x_3 & \qquad\qquad + s_3 = 8 & \qquad (4)
\end{array}
$$

The optimal solution (in the book form is)

$$
\begin{array}{llllllll}
z & +5x_2 & & & +20s_2 & +10s_3 & = & 280 \qquad (5) \\
& -2x_2 & & +s_1 & +2s_2 & -8s_3 & = & 24 \qquad (6) \\
& -2x_2 & +x_3 & & +2s_2 & -4s_3 & = & 8 \qquad (7) \\
x_1 & +1.25x_2 & & & -.5s_2 & +1.5s_3 & = & 2 \qquad (8)
\end{array}
$$

Question: Suppose I tell you which variables are basic in the optimal solution. From that information, can you easily derive the optimal solution?

Answer:

From the optimal basis I can determine which variables are basic .

Basic and nonbasic variables

The optimal solution (in the book form is)

$$
\begin{array}{llllllll}
z & +5x_2 & & & +20s_2 & +10s_3 & = & 280 \\
 & -2x_2 & & +s_1 & +2s_2 & -8s_3 & = & 24 \\
 & -2x_2 & +x_3 & & +2s_2 & -4s_3 & = & 8 \\
x_1 & +1.25x_2 & & & -.5s_2 & +1.5s_3 & = & 2
\end{array}
\begin{array}{r} (9) \\ (10) \\ (11) \\ (12) \end{array}
$$

$$BV = \{s_1, x_3, x_1\}$$

and nonbasic

$$NBV = \{x_2, s_2, s_3\}.$$

For convenience, lets convert these into vectors:

$$x_{BV} = \begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} \textbf{ and } x_{NBV} = \begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix}.$$

We can now partition the original LP based on the non-basic and basic variables.

Partitioning the original LP

$$\begin{array}{llllllll} \textbf{maximize} & 60x_1 & + & 30x_2 & + & 20x_3 & & & (13) \end{array}$$

subject to

$$8x_1 + 6x_2 + x_3 + s_1 = 48 \quad (14)$$

$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \quad (15)$$

$$2x_1 + 1.5x_2 + .5x_3 + s_3 = 8 \quad (16)$$

$$x_{BV} = \begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} \textbf{ and } x_{NBV} = \begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix}.$$

We can now partition the original LP based on the non-basic and basic variables.

$$x_{BV} = \begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} \textbf{ and } x_{NBV} = \begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix}.$$

First c_{BV} **, the coefficients of the basic vars in the original obj. function.**

$$c_{BV} = (0\ 20\ 60)$$

and c_{NBV} **, the coefficients of the basic vars in the original obj. function.**

$$c_{NBV} = (30\ 0\ 0)\,.$$

We can also partition the columns of the A matrix into those corresponding to the basic variables (B) and nonbasic variables (N). Note that it is very important to keep the order of the columns consistent.

$$B = \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & .5 & 2 \end{pmatrix} \textbf{ and } N = \begin{pmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{pmatrix}.$$

Finally, we just copy the b vector

$$b = \begin{pmatrix} 48 \\ 20 \\ 8 \end{pmatrix}$$

.

Finding the optimal basis

$$x_{BV} = \begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} \; x_{NBV} = \begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix} \; c_{BV} = (0\ 20\ 60) \; c_{NBV} = (30\ 0\ 0) \; b = \begin{pmatrix} 48 \\ 20 \\ 8 \end{pmatrix}.$$

$$B = \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & .5 & 2 \end{pmatrix} \textbf{ and } N = \begin{pmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{pmatrix}.$$

We can use this to "factor" the original LP.
objective function:

$$c^T x = c_{BV} x_{BV} + c_{NBV} x_{BNV}$$

constraints:

$$Ax = Bx_{BV} + Nx_{NBV} \leq b.$$

Let's check this out numerically

$$
\begin{aligned}
c_{BV}x_{BV} + c_{NBV}x_{BNV} &= (0\ 20\ 60)\begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} + (30\ 0\ 0)\begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix} \\
&= 0s_1 + 20x_3 + 60x_1 + 30x_2 + 0s_2 + 0s_3 \\
&= 60x_1 + 30x_2 + 20x_3
\end{aligned}
$$

Similarly

$$
Ax = Bx_{BV} + Nx_{NBV} = \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & .5 & 2 \end{pmatrix}\begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix} = \textbf{ verify on blackboard}.
$$

Finding the optimal basis

So we can write any LP as

$$
\begin{aligned}
z = \; & c_{BV}x_{BV} + c_{NBV}x_{NBV} & \\
& Bx_{BV} + Nx_{NBV} & = b \\
& x_{BV}, x_{NBV} & \geq 0
\end{aligned}
$$

Let's take the constraints and multiply by B^{-1}.

$$
\begin{aligned}
& Bx_{BV} + Nx_{NBV} &= b \quad (17)\\
\Rightarrow \; & B^{-1}(Bx_{BV} + Nx_{NBV}) &= B^{-1}b \quad (18)\\
\Rightarrow \; & x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \quad (19)\\
\Rightarrow \; & x_{BV} &= B^{-1}b - B^{-1}Nx_{NBV} \quad (20)
\end{aligned}
$$

So we can solve for x_{BV} in terms of B, N, b and x_{NBV}.

Now let's look at the objective function, and substitute for x_{BV} from equation 20:

$$
\begin{aligned}
z &= c_{BV}x_{BV} + c_{NBV}x_{NBV} \\
&= c_{BV}(B^{-1}b - B^{-1}Nx_{NBV}) + c_{NBV}x_{NBV} \\
&= c_{BV}B^{-1}b - (c_{BV}B^{-1}N - c_{NBV})x_{NBV}
\end{aligned}
$$

So we can solve for z in terms of B, N, b and x_{NBV}.

Checking with our example

$$x_{BV} = B^{-1}b - B^{-1}Nx_{NBV}$$

$$B^{-1} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -.5 & 15 \end{pmatrix}$$

$$B^{-1}b = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -.5 & 15 \end{pmatrix} \begin{pmatrix} 48 \\ 20 \\ 8 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 2 \end{pmatrix}.$$

$$B^{-1}N = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -.5 & 15 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 1.25 & -.5 & 1.5 \end{pmatrix}$$

Putting it together:

$$\begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 1.25 & -.5 & 1.5 \end{pmatrix} \begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix}$$

Compare with

$$
\begin{array}{lllllllll}
z & & +5x_2 & & & +20s_2 & +10s_3 & = & 280 \\
& & -2x_2 & & +s_1 & +2s_2 & -8s_3 & = & 24 \\
& & -2x_2 & +x_3 & & +2s_2 & -4s_3 & = & 8 \\
& x_1 & +1.25x_2 & & & -.5s_2 & +1.5s_3 & = & 2
\end{array}
\tag{21--24}
$$

And the objective function is

$$
\begin{aligned}
z &= c_{BV}B^{-1}b - (c_{BV}B^{-1}N - c_{NBV})x_{NBV} \\
&= (0\ 20\ 60)\begin{pmatrix} 24 \\ 8 \\ 2 \end{pmatrix} - \left((0\ 20\ 60)\begin{pmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 1.25 & -.5 & 1.5 \end{pmatrix} - (30\ 0\ 0) \right)\begin{pmatrix} x_2 \\ s_2 \\ s_3 \end{pmatrix} \\
&= 180 - 5x_2 - 10s_2 - 10s_3
\end{aligned}
$$

Summary of some formulas

- **Let a_j be the column corresponding to x_j in A.**
- **Let $\bar{c}_j$ be the coefficient of x_j in the optimal tableaux.**
- $\bar{c}_j = c_{BV} B^{-1} a_j - c_j$

We can simplify last formula if x_j is slack or excess or artificial:

- **If x_j is slack, $c_j = 0$ and a_j is a vector with one 1 and the rest 0's, so** $\bar{c}_j = c_{BV} B^{-1} a_j = j$**th element of**$c_{BV} B^{-1}$
- **If x_j is excess, $c_j = 0$ and a_j is a vector with one -1 and the rest 0's, so** $\bar{c}_j = c_{BV} B^{-1} a_j = -i$**th element of**$c_{BV} B^{-1}$
- **If x_j is artificial, $c_j = -M$ and a_j is a vector with one 1 and the rest 0's, so** $\bar{c}_j = c_{BV} B^{-1} a_j = i$**th element of**$c_{BV} B^{-1} + M$

Another example

$$
\begin{aligned}
z = \quad & x_1 + 4x_2 \\
& x_1 + 2x_2 + s_1 &= 6 \\
& 2x_1 + x_2 + s_2 &= 8 \\
& x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}
$$

Suppose that in the optimal basis, the basic variable are x_2 and s_2. Then,

$$x_{BV} = \begin{pmatrix} x_2 \\ s_2 \end{pmatrix} \quad x_{NBV} = \begin{pmatrix} x_1 \\ s_1 \end{pmatrix}$$

$$c_{BV} = (4\ 0) \quad x_{NBV} = (1\ 0)$$

$$B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 8 \end{pmatrix}.$$

We compute

$$B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}.$$

- **The optimal final tableaux can be computed from the formulas:**

$$
\begin{aligned}
z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\
x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b
\end{aligned}
$$

Computing the optimal basis

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

We compute

$$B^{-1}N = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}.$$

$$B^{-1}b = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$c_{BV}(B^{-1}N) = (4\ 0) \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} = (2\ 2)$$

$$c_{BV}(B^{-1}b) = (4\ 0) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 12$$

We now have the objective function

$$z + ((2\ 2) - (1\ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

and the constraints

$$\begin{pmatrix} x_2 \\ s_2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Final tableaux

We now have the objective function

$$z + ((2\ 2) - (1\ 0)) \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

and the constraints

$$\begin{pmatrix} x_2 \\ s_2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

which yields

$$\begin{aligned} z + x_1 + 2s_1 &= 12 \\ (1/2)x_1 + x_2 + (1/2)s_1 &= 3 \\ (3/2)x_1 - (1/2)s_1 + s_2 &= 5 \end{aligned}$$

Changes in parameters

- **Let's explore changes in right hand side and objective function coefficient.**
- **We will again ask when current basis remains optimal.**
  - **non-negative r.h.s.,**
  - **non-negative coefficient in row 0**
- **We proceed by example**

Change in objective function coefficient of a nbv

Let's change the objective function coefficient of x_1. x_1 is non-basic, so we are changing only c_{NBV}.
It become $(1 + \Delta\ 0)$. Looking at equations:

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

we see that c_{NBV} only changes objective function.
We recompute objective function. Instead of

$$z + ((2\ 2) - (1\ 0))\begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

we have

$$z + ((2\ 2) - (1 + \Delta\ 0))\begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = 12$$

or

$$z + (1 - \Delta)x_1 + 2s_1 = 12$$

.

So, to maintain current basis, we have $\Delta \leq 1$. Note that since x_1 is nonbasic, this is reduced cost, the amount we have to change the objective function in order to make the variable basic.

Changing the coefficient of a basic variable

Let's change the coefficient of x_2

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

Again, we only have a change in objective function. A change to c_{BV} affects the objective function in two places, so we substitute $4+\Delta$ for 4 as the first entry in c_{BV} and obtain:

$$z + \left((4+\Delta\ 0)\begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} - (1\ 0)\right)\begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = (4+\Delta\ 0)\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

or

$$z + (1+\Delta/2)x_1 + (2+\Delta/2)s_1 = 12 + 3\Delta$$

So, to keep the objective function coefficients non-negative, we must have $\Delta \geq -2$ and that each increase of the coefficient by 1 increases the objective function by 3.

Right hand side change

A change to the right hand side changes b. Let's try to change 6 to $6+\Delta$.

$$\begin{aligned} z + (c_{BV}B^{-1}N - c_{NBV})x_{NBV} &= c_{BV}B^{-1}b \\ x_{BV} + B^{-1}Nx_{NBV} &= B^{-1}b \end{aligned}$$

Note that b only appears on the right hand side.

Changing b to $\begin{pmatrix} 6+\Delta \\ 8 \end{pmatrix}$ changes $B^{-1}b$ to $\begin{pmatrix} 3+\Delta/2 \\ 5-\Delta/2 \end{pmatrix}$. Thus the system of equations becomes

$$\begin{aligned} (1/2)x_1 + x_2 + (1/2)s_1 &= 3+\Delta/2 \\ (3/2)x_1 - (1/2)s_1 + s_2 &= 5-\Delta/2 \end{aligned}$$

and the objective function value is now

$$c_{BV}B^{-1}b = (4\ 0)\begin{pmatrix} 3+\Delta/2 \\ 5-\Delta/2 \end{pmatrix} = 12 + 2\Delta$$

So we can change Δ in the range $-6 \le \Delta \le 10$ and the shadow price is 2.

Right hand side change

What about changing the 8 to $8+\Delta$. Now

$$B^{-1}b = \begin{pmatrix} 3 \\ 5+\Delta \end{pmatrix}.$$

Our allowed range of change is $-5 \leq \Delta \leq \infty$.
What about the effect on the objective function?
We compute

$$c_{BV}B^{-1}b = (4\ 0)\begin{pmatrix} 3 \\ 5+\Delta \end{pmatrix} = 12$$

So changing the right hand side of the second equation has no effect. This is not surprising, since s_2 is basic, meaning that there is slack in the second equation.