

# Gittins Index for Scheduling

## Setting

- Problem, one machine, minimize discounted weighted completion time, discount factor  $\beta$
- Jobs:
  - Distributions on processing time,  $p_j(k)$  is the probability that job  $j$  takes  $k$  units
  - weight  $w_j$
  - if job  $j$  completes at time  $t$ , the payoff is  $w_j\beta^t$ .
- Objective:  $\max E(\sum w_j\beta^{C_j})$ .

**Example:**

$j$	$p_j(1)$	$p_j(2)$	$p_j(3)$	$w_j$
1	1/2	1/2	0	8
2	1/4	0	3/4	1
3	2/3	1/6	1/6	3

What order do we run the jobs in ?

## Rule for Scheduling

- Let  $x_j$  be the amount of time that job  $j$  has already been processed.
- Let  $X_j$  be a r.v. for the amount of time that job  $j$  has already been processed.
- Compute the Gittens index of each job by formula below.
- Run the job with highest Gittens index and repeat.

$$G_j(x_j) = \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s P(X_j \geq x_j + s)}$$

## Calculations, $\beta = 1/2$

**Example:**

$j$	$p_j(1)$	$p_j(2)$	$p_j(3)$	$w_j$
<b>1</b>	<b>1/2</b>	<b>1/2</b>	<b>0</b>	<b>8</b>
<b>2</b>	<b>1/4</b>	<b>0</b>	<b>3/4</b>	<b>1</b>
<b>3</b>	<b>2/3</b>	<b>1/6</b>	<b>1/6</b>	<b>3</b>

$$G_j(x_j) = \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s (X_j \geq x_j + s)}$$

$$\Pr(X_1 = 1) = 1/2$$

$$\Pr(X_1 = 2) = 1/2$$

$$\Pr(X_1 \geq 1) = 1$$

$$\Pr(X_1 \geq 2) = 1/2$$

$$\begin{aligned} G_1(0) &= \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} (1/2)^s 8 P(X_1 = s)}{\sum_{s=1}^{\tau} (1/2)^s (X_1 \geq s)} \\ &= \max\left(\frac{1/2 \cdot 8 \cdot 1/2}{1/2 \cdot 1}, \frac{1/2 \cdot 8 \cdot 1/2 + 1/4 \cdot 8 \cdot 1/2}{1/2 \cdot 1 + 1/4 \cdot 1/2}\right) \\ &= \max(4, 3/(5/8)) \\ &= 24/5 \end{aligned}$$

## Calculations, $\beta = 1/2$

**Example:**

$j$	$p_j(1)$	$p_j(2)$	$p_j(3)$	$w_j$
<b>1</b>	<b>1/2</b>	<b>1/2</b>	<b>0</b>	<b>8</b>
<b>2</b>	<b>1/4</b>	<b>0</b>	<b>3/4</b>	<b>1</b>
<b>3</b>	<b>2/3</b>	<b>1/6</b>	<b>1/6</b>	<b>3</b>

$$G_j(x_j) = \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s (X_j \geq x_j + s)}$$

$$\Pr(X_2 = 1) = 1/4$$

$$\Pr(X_2 = 3) = 3/4$$

$$\Pr(X_2 \geq 1) = 1$$

$$\Pr(X_2 \geq 2) = 3/4$$

$$\Pr(X_2 \geq 3) = 3/4$$

$$\begin{aligned} G_2(0) &= \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} (1/2)^s 1 P(X_2 = s)}{\sum_{s=1}^{\tau} (1/2)^s (X_2 \geq s)} \\ &= \max\left(\frac{1/2 \cdot 1 \cdot 1/4}{1/2 \cdot 1}, \frac{1/2 \cdot 1 \cdot 1/4 + 1/8 \cdot 1 \cdot 3/4}{1/2 \cdot 1 + 1/8 \cdot 3/4}\right) \\ &= \max(1/4, 7/19) \\ &= 7/19 \end{aligned}$$

## Calculations, $\beta = 1/2$

**Example:**

$j$	$p_j(1)$	$p_j(2)$	$p_j(3)$	$w_j$
<b>1</b>	<b>1/2</b>	<b>1/2</b>	<b>0</b>	<b>8</b>
<b>2</b>	<b>1/4</b>	<b>0</b>	<b>3/4</b>	<b>1</b>
<b>3</b>	<b>2/3</b>	<b>1/6</b>	<b>1/6</b>	<b>3</b>

$$G_j(x_j) = \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s (X_j \geq x_j + s)}$$

$$\Pr(X_3 = 1) = 2/3$$

$$\Pr(X_3 = 2) = 1/6$$

$$\Pr(X_3 = 3) = 1/6$$

$$\Pr(X_3 \geq 1) = 1$$

$$\Pr(X_3 \geq 2) = 1/3$$

$$\Pr(X_3 \geq 3) = 1/6$$

$$\begin{aligned} G_3(0) &= \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} (1/2)^s 3 P(X_3 = s)}{\sum_{s=1}^{\tau} (1/2)^s (X_3 \geq s)} \\ &= \max\left(\frac{1/2 \cdot 3 \cdot 2/3}{1/2 \cdot 1}, \frac{1/2 \cdot 3 \cdot 2/3 + 1/4 \cdot 3 \cdot 1/6}{1/2 \cdot 1 + 1/4 \cdot 1/3}, \frac{1/2 \cdot 3 \cdot 2/3 + 1/4 \cdot 3 \cdot 1/6 + 1/8 \cdot 3 \cdot 1/6}{1/2 \cdot 1 + 1/4 \cdot 1/3 + 1/8 \cdot 1/6}\right) \\ &= \max(2, 27/14, 57/29) \\ &= 2 \end{aligned}$$

## Summary:

$$G_1(0) = 24/5$$

$$G_2(0) = 7/19$$

$$G_3(0) = 2$$

- Choose job 1.
- The max came for job 1 running for 2 units, so we run it to completion.
- Next we look at 2 and 3. In principal we could recaluculate, but it won't change anything here. So we choose job 3. Job 3's max comes from running for 1 unit. So we only run for 1 unit and then recalculate.
- $G_3(1) = \max(3/2, 9/5)$ , so we run 3 to completion.
- Now run 2.

## More general setting

We could have payoffs at each time period:  $w_j(x_j)$  . Gittins index is now:

$$G_j(x_j) = \max_{\tau \geq 0} \frac{E(\sum_{s=0}^{\tau-1} \beta^{s+1} w_j(X_j(s)) | X_j(0) = x_j)}{E(\sum_{s=0}^{\tau-1} \beta^{s+1} | X_j(0) = x_j)}$$