Gittins Index for Scheduling

Setting

- **Problem, one machine, minimize discouted weighted completion time, discount factor β**
- **Jobs:**
  - **Distributions on processing time, $p_j(k)$ is the probability that job j takes k units**
  - **weight w_j**
  - **if job j completes at time t, the payoff is $w_j\beta^t$.**
- **Objective: $\max E(\sum w_j \beta^{C_j})$.**

Example:

j	$p_j(1)$	$p_j(2)$	$p_j(3)$	w_j
1	1/2	1/2	0	8
2	1/4	0	3/4	1
3	2/3	1/6	1/6	3

What order do we run the jobs in ?

Rule for Scheduling

- **Let x_j be the amount of time that job j has already been processed.**
- **Let X_j be a r.v. for the amount of time that job j has already been processed.**
- **Compute the Gittens index of each job by formula below.**
- **Run the job with highest Gittens index and repeat.**

$$G_j(x_j) = \max_{\tau > 0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s P(X_j \geq x_j + s)}$$

Calculations, $\beta = 1/2$

Example:

j	$p_j(1)$	$p_j(2)$	$p_j(3)$	w_j
1	**1/2**	**1/2**	**0**	**8**
2	**1/4**	**0**	**3/4**	**1**
3	**2/3**	**1/6**	**1/6**	**3**

$$G_j(x_j) = \max_{\tau>0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s P(X_j \geq x_j + s)}$$

$$\begin{aligned}
\Pr(X_1 = 1) &= 1/2 \\
\Pr(X_1 = 2) &= 1/2 \\
\Pr(X_1 \geq 1) &= 1 \\
\Pr(X_1 \geq 2) &= 1/2 \\
G_1(0) &= \max_{\tau>0} \frac{\sum_{s=1}^{\tau}(1/2)^s 8P(X_1 = s)}{\sum_{s=1}^{\tau}(1/2)^s P(X_1 \geq s)} \\
&= \max(\frac{1/2 \cdot 8 \cdot 1/2}{1/2 \cdot 1}, \frac{1/2 \cdot 8 \cdot 1/2 + 1/4 \cdot 8/cdot1/2}{1/2 \cdot 1 + 1/4 \cdot 1/2}) \\
&= \max(4, 3/(5/8)) \\
&= 24/5
\end{aligned}$$

Calculations, $\beta = 1/2$

Example:

j	$p_j(1)$	$p_j(2)$	$p_j(3)$	w_j
1	**1/2**	**1/2**	**0**	**8**
2	**1/4**	**0**	**3/4**	**1**
3	**2/3**	**1/6**	**1/6**	**3**

$$G_j(x_j) = \max_{\tau>0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s P(X_j \geq x_j + s)}$$

$$\begin{aligned}
\Pr(X_2 = 1) &= 1/4 \\
\Pr(X_2 = 3) &= 3/4 \\
\Pr(X_2 \geq 1) &= 1 \\
\Pr(X_2 \geq 2) &= 3/4 \\
\Pr(X_2 \geq 3) &= 3/4 \\
G_2(0) &= \max_{\tau>0} \frac{\sum_{s=1}^{\tau}(1/2)^s 1 P(X_2 = s)}{\sum_{s=1}^{\tau}(1/2)^s P(X_2 \geq s)} \\
&= \max\left(\frac{1/2 \cdot 1 \cdot 1/4}{1/2 \cdot 1}, \frac{1/2 \cdot 1 \cdot 1/4 + 1/8 \cdot 1 \cdot 3/4}{1/2 \cdot 1 + 1/8 \cdot 3/4}\right) \\
&= \max(1/4, 7/19) \\
&= 7/19
\end{aligned}$$

Calculations, $\beta = 1/2$

Example:

j	$p_j(1)$	$p_j(2)$	$p_j(3)$	w_j
1	**1/2**	**1/2**	**0**	**8**
2	**1/4**	**0**	**3/4**	**1**
3	**2/3**	**1/6**	**1/6**	**3**

$$G_j(x_j) = \max_{\tau>0} \frac{\sum_{s=1}^{\tau} \beta^s w_j P(X_j = x_j + s)}{\sum_{s=1}^{\tau} \beta^s P(X_j \geq x_j + s)}$$

$$\begin{aligned}
\Pr(X_3 = 1) &= 2/3 \\
\Pr(X_3 = 2) &= 1/6 \\
\Pr(X_3 = 3) &= 1/6 \\
\Pr(X_3 \geq 1) &= 1 \\
\Pr(X_3 \geq 2) &= 1/3 \\
\Pr(X_3 \geq 3) &= 1/6 \\
G_3(0) &= \max_{\tau>0} \frac{\sum_{s=1}^{\tau}(1/2)^s 3P(X_3 = s)}{\sum_{s=1}^{\tau}(1/2)^s P(X_3 \geq s)} \\
&= \max(\frac{1/2 \cdot 3 \cdot 2/3}{1/2 \cdot 1}, \frac{1/2 \cdot 3 \cdot 2/3 + 1/4 \cdot 3 \cdot 1/6}{1/2 \cdot 1 + 1/4 \cdot 1/3}, \frac{1/2 \cdot 3 \cdot 2/3 + 1/4 \cdot 3 \cdot 1/6 + 1/8 \cdot 3 \cdot 1/}{1/2 \cdot 1 + 1/4 \cdot 1/3 + 1/8 \cdot 1/6} \\
&= \max(2, 27/14, 57/29) \\
&= 2
\end{aligned}$$

Summary:

$$
\begin{aligned}
G_1(0) &= 24/5 \\
G_2(0) &= 7/19 \\
G_3(0) &= 2
\end{aligned}
$$

- **Choose job 1.**
- **The max came for job 1 running for 2 units, so we run it to completion.**
- **Next we look at 2 and 3. In principal we could recaluculate, but it won't change anything here. So we choose job 3. Job 3's max comes from running for 1 unit. So we only run for 1 unit and then recalculate.**
- $G_3(1) = \max(3/2, 9/5)$**, so we run 3 to completion.**
- **Now run 2.**

More general setting

We could have payoffs at each time period: $w_j(x_j)$ **. Gittins index is now:**

$$G_j(x_j) = \max_{\tau \geq 0} \frac{E(\sum_{s=0}^{\tau-1} \beta^{s+1} w_j P(X_j(s)) | X_j(0) = x_j)}{E(\sum_{s=0}^{\tau-1} \beta^{s+1} P(X_j(0) = x_j)}$$