Reductions

Reduction: Problem A reduces to Problem B if, given a “black box” (subroutine) for B, one can solve A using a (polynomial) number of calls to the subroutine.

Trivial Example:

- B is addition – $B(x, y) = x + y$
- A multiplication by 3.
- A reduces to B because we can multiply by 3: $A(z) = B(z, B(z, z))$. 
More Reduction Examples

• A is max flow, B is linear programming
• A is $1||\sum C_j$, B is $1||\sum w_jC_j$
• A is $P||C_{\text{max}}$, B is $P|\text{prec}|\sum w_jC_j$
Reductions for NP-completeness

• For technical reasons, we will only consider decision versions of problems.

• e.g. $P||C_{\text{max}}$; Given $m$ machines, $n$ jobs and a number $B$, does the optimal schedule have makespan less than $B$.

• e.g. Shortest Paths: Given a graph $G$ with weights on the edges, two distinguished vertices $s$ and $t$ and a number $B$, is the shortest path from $s$ to $t$ of length less than $B$.

• The decision version and the optimization version of a problem are “equivalent,” that is they each reduce to each other.
**Reduction Example**

**Vertex Cover**  A vertex cover of a graph \( G=(V,E) \) is a set of vertices \( V' \), such that for every edge \( (x,y) \), at least one of \( x \) and \( y \) is in \( V' \). The vertex cover problem is given a graph \( G \) and a number \( k \) and asks whether \( G \) has a vertex of size at most \( k \).

**Clique**  A clique is a set of vertices such that each pair of vertices has an edge between them. The clique problem is given a graph and a number \( ℓ \) and asks when a graph has a clique of size at least \( ℓ \).

**Question:** Show that vertex cover reduces to clique.