Definition of a Linear Program

**Definition:** A function $f(x_1, x_2, \ldots, x_n)$ of $x_1, x_2, \ldots, x_n$ is a **linear function** if and only if for some set of constants $c_1, c_2, \ldots, c_n$,

$$f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$ 

**Examples:**

- $x_1$
- $5x_1 + 6x_4 - 2x_2 + 1$
- $3$

**Non-examples:**

- $x_1^2$
- $x_1 + 3x_2 - 4x_3^4$
- $x_1 x_2$
Linear Inequalities

**Definition:** For any linear function \( f(x_1, x_2, \ldots, x_n) \) and any number \( b \), the inequalities

\[
f(x_1, x_2, \ldots, x_n) \leq b
\]

and

\[
f(x_1, x_2, \ldots, x_n) \geq b
\]

are linear inequalities.

**Examples:**

- \( x_1 + x_2 \leq 4 \)
- \( 5x_1 - 4 \geq 0 \)

**Note:** If an inequality can be rewritten as a linear inequality then it is one. Thus \( x_1 + x_2 \leq 3x_3 \) is a linear inequality because it can be rewritten as \( x_1 + x_2 - 3x_3 \leq 0 \). **Even** \( x_1/x_2 \leq 4 \) is a linear inequality because it can be rewritten as \( x_1 - 4x_2 \leq 0 \). **Note that** \( x_1/x_2 + x_3 \leq 4 \) is not a linear inequality, however.

**Definition:** For any linear function \( f(x_1, x_2, \ldots, x_n) \) and any number \( b \), the equality

\[
f(x_1, x_2, \ldots, x_n) = b
\]

is a linear equality.
**LPs**

**Definition:** A linear programming problem (LP) is an optimization problem for which:

1. We attempt to maximize (or minimize) a linear function of the decision variables. *(objective function)*

2. The values of the decision variables must satisfy a set of constraints, each of which must be a linear inequality or linear equality.

3. A sign restriction on each variable. For each variable $x_i$ the sign restriction can either say
   
   (a) $x_i \geq 0$,
   
   (b) $x_i \leq 0$,
   
   (c) $x_i$ unrestricted (urs).

**Definition:** A solution to a linear program is a setting of the variables.

**Definition:** A feasible solution to a linear program is a solution that satisfies all constraints.

**Definition:** The feasible region in a linear program is the set of all possible feasible solutions.
Definition: A set of points $S$ is a **convex set** if the line segment joining any 2 points in $S$ is wholly contained in $S$.

Fact: The set of feasible solutions to an LP (feasible region) forms a (possibly unbounded) convex set.

Definition: A point $p$ of a convex set $S$ is an **extreme point** if each line segment that lies completely in $S$ and contains $p$ has $p$ as an endpoint. An extreme point is also called a **corner point**.

Fact: Every linear program has an extreme point that is an optimal solution.

Corrolary: An algorithm to solve a linear program only needs to consider extreme points.

Definition: A constraint of a linear program is **binding** at a point $p$ if the inequality is met with equality at $p$. It is **nonbinding** otherwise. (Recall that a point is the same as a solution.)
Solutions to Linear Programs

**Definition:** An optimal solution to a linear program is a feasible solution with the largest objective function value (for a maximization problem). The value of the objective function for the optimal solution is said to be the value of the linear program. A linear program may have multiple optimal solutions, but only one optimal solution value.

**Definition** A linear program is infeasible if it has no feasible solutions, i.e. the feasible region is empty.

**Definition** A linear program is unbounded if the optimal solution is unbounded, i.e. it is either $\infty$ or $-\infty$. Note that the feasible region may be unbounded, but this is not the same as the linear program being unbounded.

**Theorem** Every linear program either:

1. is infeasible,
2. is unbounded,
3. has a unique optimal solution value
Basic and nonbasic variables and solutions

Consider a linear programming problem in which all the constraints are equalities (conversion can be accomplished with slack and excess variables).

**Definition:** If there are \( n \) variables and \( m \) constraints, a solution with at most \( m \) non-zero values is a **basic solution**.

**Definition** In a basic solution, \( n - m \) of the zero-valued variables are considered **non-basic variables** and the remaining \( m \) variables are considered **basic variables**.

**Fact:** Any linear program that is not infeasible or unbounded has an optimal solution that is basic. Basic feasible solutions correspond to extreme points of the feasible region.

**Fact:** Let \( B \) be formed from \( m \) linearly independent columns of the \( A \) matrix (basis), and let \( x_B \) be the corresponding variables, then \( Bx_B = b \).
Some facts to prove on the board:

- The feasible region of an LP is convex.
- If an LP, \( \max_x c^T x \text{ s.t. } Ax = b, x \geq 0 \) has a feasible solution, it has a basic feasible solution.
- \( x \) is an extreme point of \( S = \{x : Ax = b, x \geq 0\} \) iff \( x \) is a basic feasible solution.
- If an LP has a finite optimal solution, then it has an optimal solution at an extreme point of the feasible set.
- If there is an optimal solution to an LP problem, then there is an optimal basic feasible solution.
Simplex

- The simplex algorithm moves from basic feasible solution to basic feasible solution; at each iteration it increases (does not decrease) the objective function value.

- Each step is a pivot, it chooses a variable to leave the basis, and another to enter the basis. The entering variable is a non-basic variable with positive objective function coefficient. The leaving variable is chosen via a ratio test, and is the basic variable in the constraint that most limits the increase.

- When the objective function has no negative coefficients in the objective function, we can stop.
Simplex Example

Original LP

maximize  \[ 3x_1 + x_2 + 2x_3 \]  \hspace{1cm} (1)
subject to
\[ x_1 + x_2 + 3x_3 \leq 30 \]  \hspace{1cm} (2)
\[ 2x_1 + 2x_2 + 5x_3 \leq 24 \]  \hspace{1cm} (3)
\[ 4x_1 + x_2 + 2x_3 \leq 36 \]  \hspace{1cm} (4)
\[ x_1, x_2, x_3 \geq 0 . \]  \hspace{1cm} (5)

Standard form.

\[ z = 3x_1 + x_2 + 2x_3 \]  \hspace{1cm} (6)
\[ x_4 = 30 - x_1 - x_2 - 3x_3 \]  \hspace{1cm} (7)
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \]  \hspace{1cm} (8)
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 . \]  \hspace{1cm} (9)
Iteration 1

\[ z = 3x_1 + x_2 + 2x_3 \quad (10) \]
\[ x_4 = 30 - x_1 - x_2 - 3x_3 \quad (11) \]
\[ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \quad (12) \]
\[ x_6 = 36 - 4x_1 - x_2 - 2x_3 \quad (13) \]

Pivot in \( x_1 \). Remove \( x_6 \) from the basis.

\[ z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \quad (14) \]
\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \quad (15) \]
\[ x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \quad (16) \]
\[ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \quad (17) \]
Iteration 2

\[ z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \]  
(18)

\[ x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \]  
(19)

\[ x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \]  
(20)

\[ x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \]  
(21)

Pivot in \( x_3 \). Remove \( x_5 \).

\[ z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \]  
(22)

\[ x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \]  
(23)

\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]  
(24)

\[ x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \]  
(25)
Iteration 3

\[ z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \]  \hspace{1cm} (26)

\[ x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \]  \hspace{1cm} (27)

\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]  \hspace{1cm} (28)

\[ x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}. \]  \hspace{1cm} (29)

**Pivot in** \( x_2 \). **Remove** \( x_3 \).

\[ z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \]  \hspace{1cm} (30)

\[ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \]  \hspace{1cm} (31)

\[ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \]  \hspace{1cm} (32)

\[ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}. \]  \hspace{1cm} (33)
Duality

Given a linear program (primal)

\[
\begin{align*}
\text{max } c^T x \\
Ax & \leq b \\
x & \geq 0
\end{align*}
\]

The dual is

\[
\begin{align*}
\text{min } b^T y \\
A^T y & \geq c \\
y & \geq 0
\end{align*}
\]

\(x\) is an \(n\)-dimensional vector
\(y\) is an \(m\)-dimensional vector
\(c\) is an \(n\)-dimensional vector
\(b\) is an \(m\)-dimensional vector
\(A\) is a \(m \times n\) matrix
**Dual Theorem**

**Theorem**  Given a primal and dual linear program, let $x^*$ be the optimal solution to the primal and $y^*$ be the optimal solution to the dual. Then

$$c^T x^* = b^T y^*.$$  

Furthermore, the optimal dual solution can be read off as the negative of the coefficients of the corresponding slack/excess variables in the final simplex tableaux.

**Proof:**  Denote the constraints of the original LP as:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j$$

where $x_{n+i}$ are the original slack variables.

Solve using simplex. The final objective function is of the form:

$$Z = Z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k$$

where $\bar{c}_k$ are the coefficients of the basic variables and zero for nonbasic variables. Thus, $\bar{c}_k \leq 0$ because the conditions for stopping is that of slack variables be negative and the basic variables equal zero.
Proof continued

So we have

\[ Z = Z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k \]

From this equation it is easy to find the dual solution by reading the coefficients of the slack variables.

Let \( y_i^* = -\bar{c}_{n+i} \). The claim is that \( y_i^* \) satisfies

\[ \sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \]

\[ \sum_{j=1}^{n} c_j x_j^* = Z \]

\[ = Z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k \]

\[ = Z^* + \sum_{j=1}^{n} (\bar{c}_j x_j) + \sum_{i=1}^{m} (\bar{c}_{n+i} x_{n+i}) \]

\[ = Z^* + \sum_{j=1}^{n} (\bar{c}_j x_j) - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j) \]

The last line follows from the definition of \( y_i^* \) and the definitions of the slack variables in the original LP.
Proof continued

So we have that the final objective function can be written as

\[ Z^* + \sum_{j=1}^{n} (\bar{c}_j x_j) - \sum_{i=1}^{m} y_i^*(b_i - \sum_{j=1}^{n} a_{ij} x_j). \]

Reordering terms, this is

\[ (Z^* - \sum_{i=1}^{m} b_i y_i^*) + \sum_{j=1}^{n} (\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*) x_j. \]

Now, recall that pivoting is just rewriting in an equivalent way, so we actually have that, for any feasible \( x \):

\[ \sum_{j=1}^{n} c_j x_j = (Z^* - \sum_{i=1}^{m} b_i y_i^*) + \sum_{j=1}^{n} (\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*) x_j. \]

Because this equation must be true for an infinite number of \( x \) values we must have that (look at colors)

\[ (Z^* - \sum_{i=1}^{m} b_i y_i^*) = 0. \]

and

\[ c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*. \]
Proof continued

\[(Z^* - \sum_{i=1}^{m} b_i y_i^*) = 0.\]

and

\[c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*.\]

The first of these says that

\[Z^* = \sum_{i=1}^{m} b_i y_i^*\]

in other words, \(y^*\) is optimal.

The second says that

\[\sum_{i=1}^{m} a_{ij} y_i^* = c_j - \bar{c}_j\]

But recall that we have said that \(\bar{c}_j \leq 0\) because these are coefficients of the objective function in the final tableaux. Therefore,

\[\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j\]

and \(Y^*\) is feasible.
Complimentary Slackness

**Theorem**  Let $x^*$ be feasible in $P$ and let $y^*$ be feasible in $D$. Necessary and sufficient conditions for optimality of $x^*$ and $y^*$ are

$$
\sum_{i=1}^{m} a_{ij} y_i^* = c_j \ OR \ x_j^* = 0 \quad \text{for all } j = 1, \ldots, n
$$

AND

$$
\sum_{j=1}^{n} a_{ij} x_j^* = b_i \ OR \ y_i^* = 0 \quad \text{for all } i = 1, \ldots, m
$$

Either constraint is tight or dual variable is zero.