Max Flow

Push/Relabel Methods

1) \( d(v) \) - "distances" (~ lower bounds on dist. to source or sink)

2) Preflow \( f \)
   1) \( 0 \leq f(vw) \leq u(vw) \quad \forall (v,w) \in E \)
   2) \( e(v) = \sum_w f(w,v) - \sum_w f(v,w) \geq 0 \quad \forall v \in V \setminus \{s,t\} \)

\[ \text{excess} \quad \text{flow in} \geq \text{flow out} \]
Intuition
- Water flow down hill
- If water accumulates at V, raise V
- Source is at fixed h_f
- Water can eventually flow back to s
- Algorithm maintains a preflow \( w/ \sum_{v} f(s, v) \geq f^+(\text{max flow value}) \)
- Aug paths maintain a flow \( w/ \sum_{v} f(s, v) \leq f^+ \)

Aug paths always feasible more towards optimality
P/R always (super) optimal more towards feasibility
If $e(v) > 0$, $v$ is active
If $u_f(v,w) > 0$ $(v,w)$ is admissible
and $d(v) = d(w) + 1$

2 operations: e=y

Push $(v,w)$
Applies when: $e(v) > 0$; $u_f(v,w) > 0$; $d(v) = d(w) + 1$
Action:
\[ \delta = \min \{ e(v), u_f(v,w) \} \]
\[ f(v,w) \uparrow = \delta \]
\[ e(v) - \delta \downarrow \]
\[ e(w) \uparrow = \delta \]

Relabel $(v)$
Applies when: $e(v) > 0$; no edge $(v,w) \in E_f$ is admissible
Action:
\[ d(v) = \min \{ d(w) \} + 1 \]
Generic Push/Relabel Alg

1) $f = 0$; $d(s) = n$; $d(t) = 0$; $d(v) = \text{s.p. distance to } t$

2) Saturate all edges out of $s$
   
   \[ f(s,v) = u(s,v); \quad e(v) = f(s,v) \]

3) While a push or relabel applies
   
   Execute it
Properties

1) \(d(v) \leq d'(v)\)

\(d(v)\) is always a lower bound on the dist to sink or source.

Inductive proof:

- After a push, you don't create any shorter paths

- relabel

\[f(v) = \min \left\{ \text{sp.d. to s in } G_f, \text{sp.d. to t in } G_f \right\}\]
Before relabel, $v$ has no outgoing admissible edges.

Properties is maintained.

$G_f$
2) \( d(v) \leq d(w) + 1 \) \( \forall (v,w) \in G_f \)

3) If \( e(v) > 0 \) then
   
   Either
   
   a) If an edge \((v,w) \in G_f\) on which you can push flow
   
   b) you can relabel \( v \).
4) If \( e(v) = 0 \) \( \forall v \in V - \{s,t\} \) then \( f \) is a maximum flow

PF

1) If \( e(v) = 0 \) \( \forall v \in V - \{s,t\} \), then \( f \) is a flow

2) There is never an \( s-t \) path in \( G_f \)
Running Time

1) distance labels

\[ 0 \leq d(v) \leq n + (n-2) \leq 2n \]

\[ \#\text{Relabels} = \mathcal{O}(n^2) \]

Time / relabel
Time to relabel $v \approx \deg(v) = O(n)$

Total time relabeling = $O(n^3)$

Total time to relabel each vertex once

$= \sum_{v} \deg(v) = 2m$

Total time to relabel each vertex $2n$ times

$\leq 2n(2m) = O(nm)$
Pushes

Saturation: sends $u_f(v, w)$

Non-saturation: sends $e(v)$

Diagram:

- $e = 4$
- $u = 3$
- $d = 10$
- $e = 4$
- $u = 5$
- $d = 10$
- $e = 0$
- $u = 1$
- $d = 10$
- $e = 0$
- $u = 1$
- $d = 9$
Satulating pushes:

Satulating push \( (v, w) \)
- relabel \( w \)
- push \( (w, v) \)
- relabel \( v \)

Saturation push \( (v, w) \)

\[ \# \text{sat pushes} (v, w) \leq 2n \]

Total \( \# \text{sat pushes} \leq (2n)(2m) = O(nm) \)
Non-saturating postes

\[ \emptyset = \sum_{v : e(v) > 0} d(v). \]
\[ \phi = \sum_{e(v) > 0} d(v). \]

After initialization, \[ \phi \leq 2n^2 \]
Finally, \[ \phi = 0 \]

Total increase from relabelling \( \leq R \)
Total "" "" sat. pushes \( \leq S \)
Each n.s. push decreases \( \phi \) by \( \geq 1 \)
Total decrease from n.s. pushes \( \leq 2n^2 + R + S \)

Total decrease from n.s. pushes \( \leq 2n^2 + R - S \)
Relabeling

\[ \theta = \sum_{e(v) > 0} d(v) \]

\[ \begin{array}{c}
\circ \quad \text{rel.} \quad \circ \\
\uparrow \quad e(v) > 0 \quad e(v) > 0 \\
10 \quad \geq 11 \\
\end{array} \]

Total increase in \( \phi \) over all relabelings \( \leq 2n^2 \)
Sat push

\[ \begin{array}{c}
\text{sat. push} \quad e \\
\text{sat. push} \quad e
\end{array} \]

\[ \begin{array}{c}
\text{sat. push} \quad e \\
\text{sat. push} \quad e
\end{array} \]

\[ \begin{array}{c}
\Phi \text{ unchanged} \\
\text{increased } \Phi \text{ by } d(w).
\end{array} \]

Total increase to \( \Phi \) from all sat. pushes \[ \leq 2n \leq (4n^3)(2n) = 8n^3 \]
non-sat. pushes

\( e \xrightarrow{\psi} e \xrightarrow{} \omega \)

\( \phi \) decreases by at least 2.

\( \phi \xrightarrow{} \omega \xrightarrow{} \omega \)

\( \phi = d(\psi) \)

\( \varphi \) decreases by \( d(\psi) \)

inc. by \( g(\omega) \)

der. by \( d(\psi) - d(\omega) \):

\( = 1 \)
Total decrease \leq \Phi_{\text{init}} - \Phi_0 + \text{inc. from rel.} \\
+ \text{inc. from str. push} \\
\leq 2n^2 \cdot 0 + 2n^2 + 8n^3 m \\
= 4n^2 + 8n^3 m \\
\therefore \# \text{str. push} = O(n^3 m)

sat push: O(n^m) \\
time rel.: O(n^m) \\
Total = O(n^2 m)
Choosing operation.
Maintain a list of $v \in \text{neighborhood}(v)$ with $e(v) > 0$. 

- $\checkmark$
- $w_1 \rightarrow s$ ← current edge

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- $u_f > 0$
- $u_f = 0$
- $u_f > 0$
- $u_f = 0$

10
3
5
8
Discharge($v$)

While $e(v) > 0$ and $v$ has not been relabeled

$w = \text{current-edge}(v)$

if $(v,w)$ is admissible

push($v,w$)

else if $(v,w)$ is not the last edge in $v$'s list

advance current-edge($v$)

else

relabel($v$)
Observation

- In one call to Discharge(v), all pushes except possibly the last are saturating.

- Non-saturating pushes cause Discharge to terminate.

- Each time the current edge pointer advances through the entire list, v is relabeled.

\[ \text{Data structure overhead} = O(nm) \]