Max flow

Push/Relabel Methods

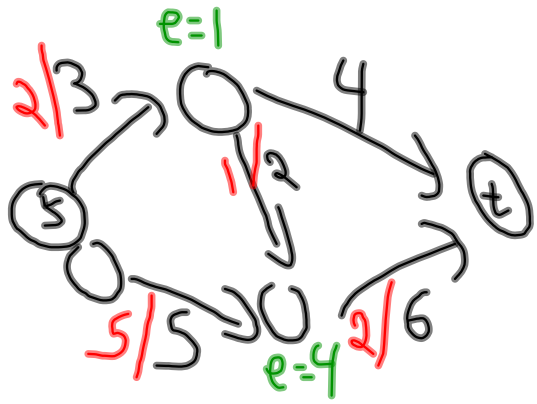
1) $d(v)$ - "distances" (~ lower bounds on dist. to source or sink)

2) preflow f

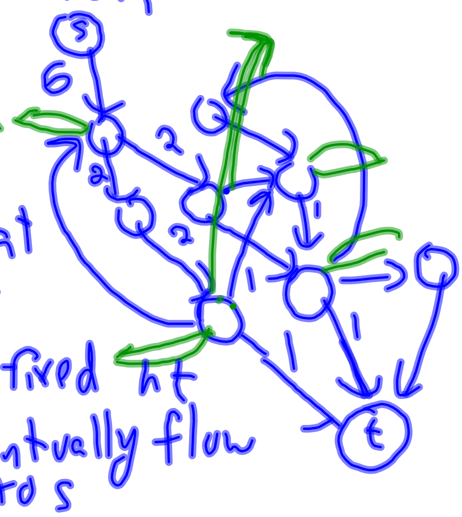
   1) $0 \le f(vw) \le u(vw) \quad \forall (v,w) \in E$

   2) $\underbrace{e(v)}_{\text{excess}} \equiv \sum_w f(w,v) - \sum_w f(v,w) \ge 0 \quad \forall v \in V - \{s,t\}$

      "flow in $\geq$ flow out"



Intuition



- water flow down hill
- if water accumulates at v, raise v
- source is at fixed ht
- water can eventually flow back to s

P/R

- Algorithm maintains a <u>preflow</u>

  w/ $\sum_v f(s,v) \geq |f^*|$ (max flow value)

- Aug paths maintain a flow, w/ $\sum_v f(s,v) \leq |f^*|$

Aug paths	always feasible	move towards optimality
P/R	always (super) optimal	move towards feasibility

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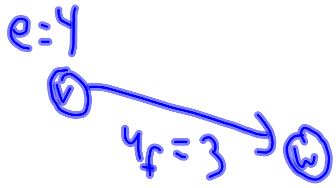
If $e(v) > 0$, v is active

If $u_f(v,w) > 0$ and $d(v) = d(w) + 1$, (v,w) is admissible



2 operations:

Push(v,w)

Applies when: $e(v)>0$; $u_f(v,w)>0$; $d(v)=d(w)+1$

Action: $\delta = \min(e(v), u_f(v,w))$

$f(v,w) \mathrel{+}= \delta$

$e(v) \mathrel{-}= \delta$

$e(w) \mathrel{+}= \delta$

Relabel(v)

Applies when: $e(v)>0$; no edge $(v,w) \in G_f$ is admissible

Action: $d(v) = \min_{(v,w)\in G_f} \{d(w)\} + 1$

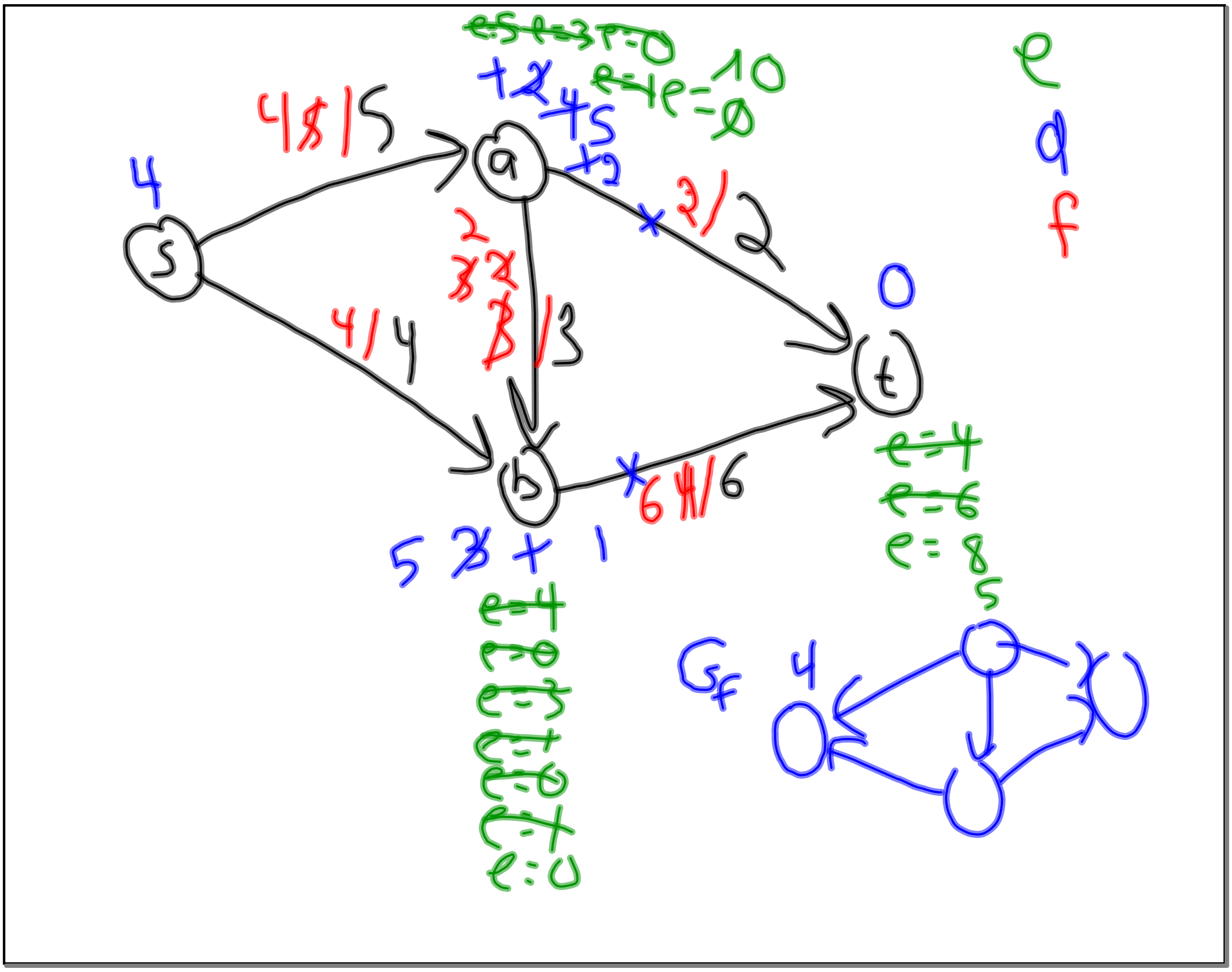
Generic Push|Relabel Alg

1) $f \equiv 0$; $d(s) = n$; $d(t) = 0$; $d(v) =$ s.p distance to t

2) Saturate all edges out of s

$$(f(s,v) = u(s,v) ; \; e(v) = f(s,v))$$

3) While a push or relabel applies
   Execute it


e
d
f
G_f

Properties

$\delta(v) = \min \left\{ \begin{array}{l} \text{s.p.d. to } s \text{ in } G_f, \\ \text{s.p.d. to } t \text{ in } G_f \end{array} \right\}$

$d(v) \le \delta(v)$

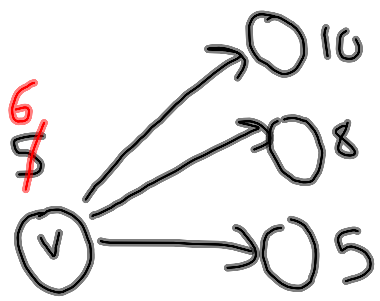
1) $d(v)$ is always a lower bound on the dist in G_f to sink or source.

Inductive proof:

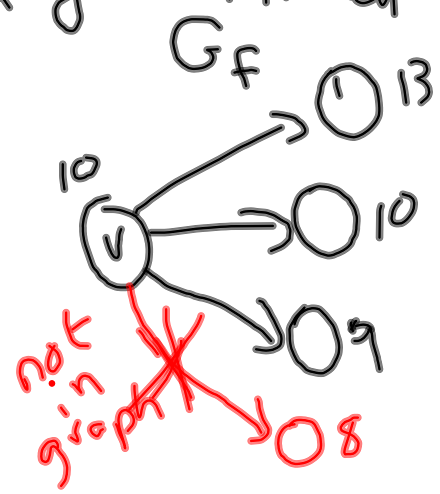
- after a push, you don't create any shorter paths

- relabel

Befor relabel, v has no outgoing admissible edges



property is maintained



2) $d(v) \le d(w) + 1 \quad \forall (v,w) \in G_f$

3) If $e(v) > 0$ then

either

a) $\exists$ an edge $(v,w) \in G_f$ on which you can push flow

b) you can relabel v.

4) If $e(v) = 0 \ \forall v \in V - \{s,t\}$ then f is a maximum flow

Pf

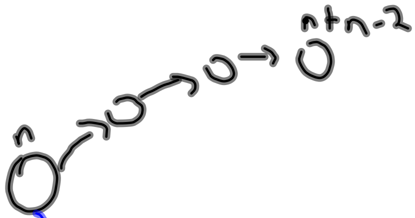
1) If $e(v) = 0 \ \forall v \in V - \{s,t\}$, then f is a flow

2) There is never an s-t path in G_f

Running Time

1) distance labels

$$0 \le d(v) \le n + (n-2) \le 2n$$



\# Relabels $= O(n^2)$

Time / relabel

Time to relabel v $\approx \deg(v) = O(n)$

Total time relabeling $= O(n^3)$

Total time to relabel each vertex once

$= \sum_v \deg(v) = 2m$

Total time to relabel each vertex 2n times

$\leq 2n(2m) = \underline{O(nm)}$

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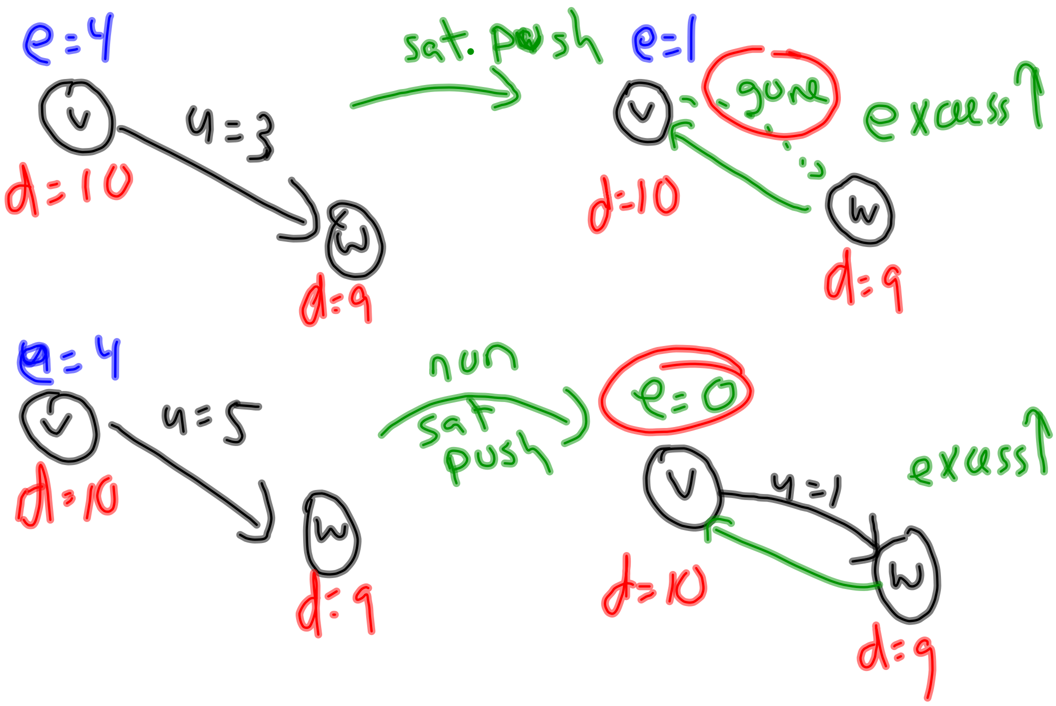
Pushes

Saturating : sends $u_f(v,w)$

Non-saturating : sends $e(v)$



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Saturating pushes:

Saturating push (v, w)

relabel w

push (w, v)

relabel v

Saturating push (v, w)

necc. events in between 2 sat. pushes

$\#$ sat pushes $(v, w) \le 2n$

total $\#$ sat pushes $\le (2n)(2m) = O(nm)$

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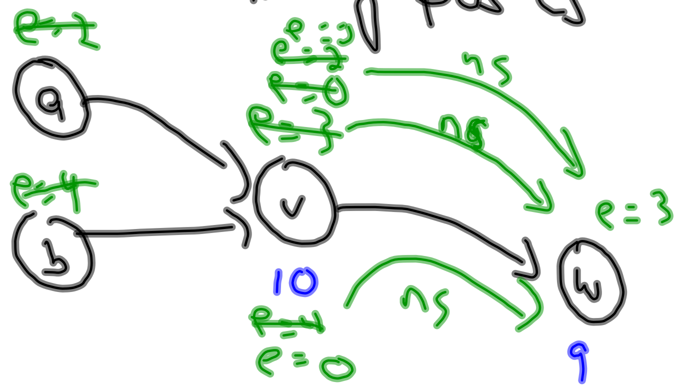
non-saturating pushes



$$\Phi = \sum_{v:e(v)>0} d(v).$$

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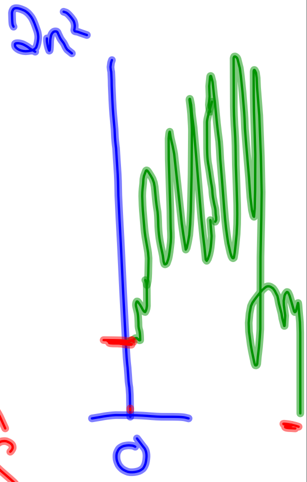
After initialization $\Phi \le 2n^2$

Finally $\Phi = 0$



Total increase from relabelling $\le R$

Total " " sat. pushes $\le S$

Each n.s. push decreases Φ by ≥ 1

Total decrease from n.s. pushes $\le 2n^2 + R + S$

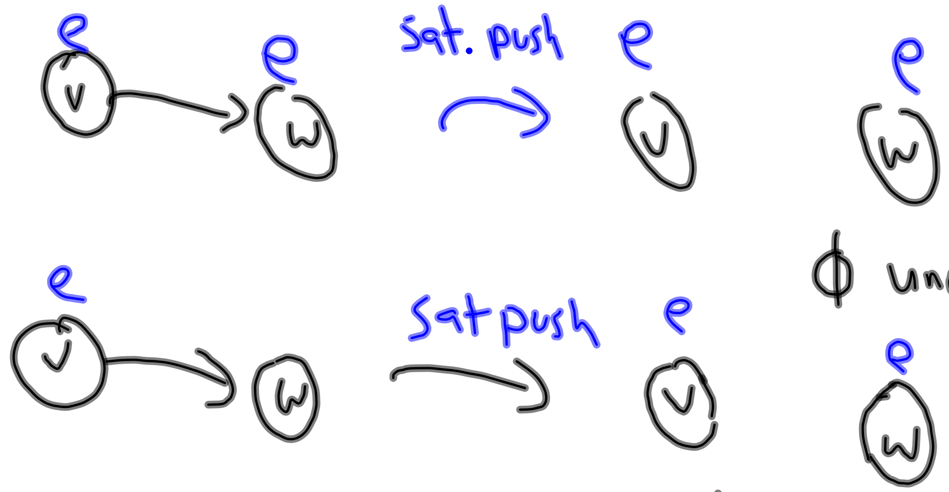
" # of " " $\le 2n^2 + R + S$

Relabeling $\quad \Phi = \sum_{e(v)>0} d(v)$

$$\underset{10}{\overset{e(v)>0}{v}} \xrightarrow{\text{rel.}} \underset{\geq 11}{\overset{e(v)>0}{v}} \qquad \text{inc. } \phi \text{ by } \geq 1$$

Total increase in ϕ over all relabelings $\leq 2n^2$

Sat push

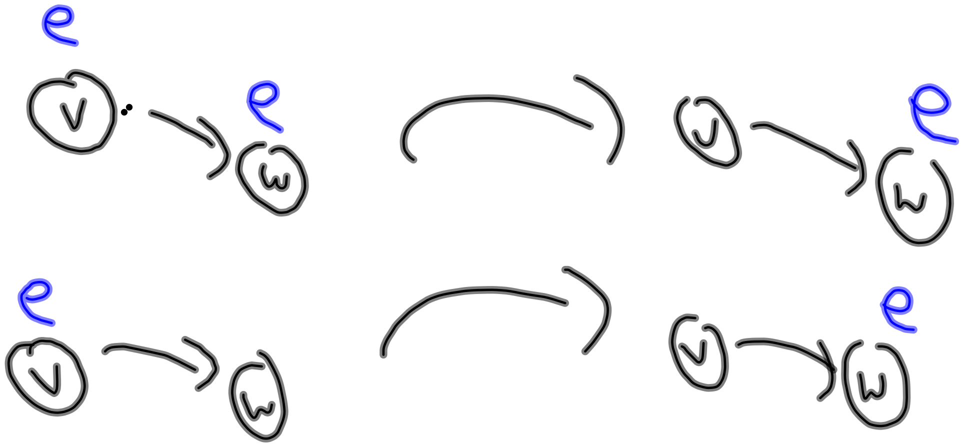


ϕ unchanged

increased ϕ by $d(w)$.

Total increase to ϕ from all sat. pushes $\leq (4nm)\underbrace{(2n)}_{\leq 2n} = 8n^2m$

non-sat. pushes



$\Phi \mathrel{-}= d(v)$

Φ decreases by at least 2!

Φ decreases by $d(v)$
inc. by $d(w)$
dec. by $d(v) - d(w)$
$= 1$

Total decrease $\le \Phi_{init} - \Phi_0$ + inc. from rel.
+ inc. from sat. pushes

$$\le 2n^2 \cdot 0 + 2n^2 + 8n^2 m$$
$$= 4n^2 + 8n^2 m$$

$\therefore$ # n.s. pushes $= O(n^2 m)$

sat pushes $O(nm)$
time rel. $O(nm)$
Total $= O(n^2 m)$

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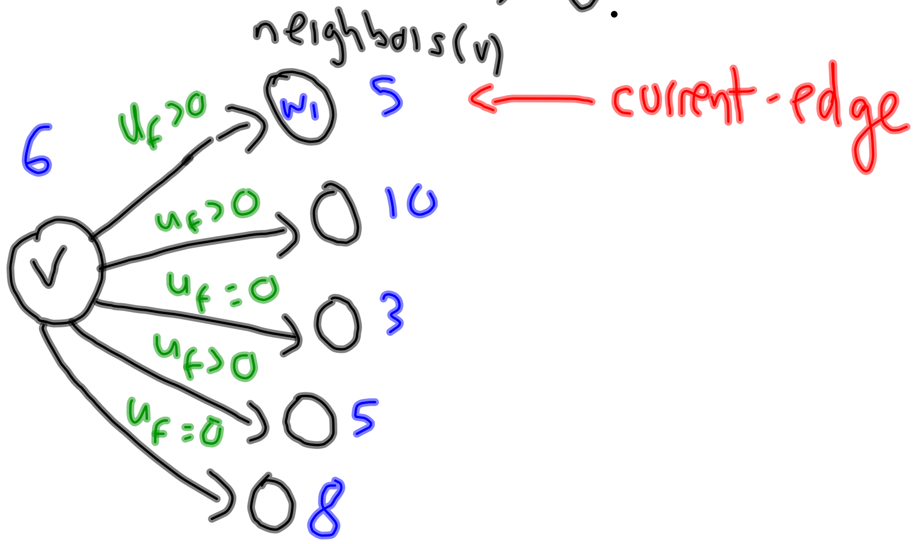
Choosing operation.

Maintain a list of v w/ $e(v) > 0$.



Discharge(v)

While e(v) > 0 and v has not been relabeled

w = current-edge(v)

if (v,w) is admissible

push (v,w)

else if (v,w) is not the last edge in v's list

advance current-edge(v)

else relabel(v)

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Observation

- In one call to Discharge(v), all pushes except possibly the last are saturating
- Nonsaturating pushes cause Discharge to terminate
- each time the current edge pointer advances through the entire list, v is relabeled

$\therefore$ time advancing v's pointer $\leq$ time spent relabeling v.

Data structure overhead $= O(nm)$

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