Choosing which vertex to call Discharge(v)

1) FIFO

Q: 2 3 4 5 6

Phase 1: nodes added to Q in initializing
Phase i: nodes added during phase i-1
Claim: The number of phases is $O(n^2)$.

Proof: \[ \Phi = \max \{ d(v) : e(v) > 0 \} \text{ on } Q \]

Look at $\Phi$ during a phase.

Case 1: A relabeling occurs during the phase.
Happens $\leq 2n^2$ times. Total increase in $\Phi$ in these phases $\leq 2n^2$.

Case 2: No relabeling occurs during the phase.

\[ e = 4 \]

\[ v_6, v_{10}, v_{12} \text{ active} \]

\[ d(v_5) = 8, d(v_{10}) = d(v_{12}) = 7 \]

Vertices with $e(v) > 0$ at the start of the phase do not have excess at the end of the phase.

No new excess at $\Phi$ or higher.

\[ \Phi \text{ decreases by at least 1.} \]
\[ \phi_{init} \leq n \]
\[ \phi_{final} = 0 \]

Total increase in \( \phi \leq 2n^2 \)

Total decrease in \( \phi \leq 2n^2 + n - 0 \leq 3n^2 \)

Each phase with a relabeling decrease \( \phi \) by \( \geq 1 \).

\[ \therefore \leq 3n^2 \text{ phases with a relabeling.} \]

\[ \therefore \leq 5n^2 \text{ phases overall.} \]
\[ \# \text{n.s. pushes per phase} \leq \eta \]

\[ \# \text{n.s. pushes per Discharge}(v) \leq 1 \]

\[ \# \text{calls to Discharge per phase} \leq \eta \]

\[ \# \text{of n.s. pushes} = \mathcal{O}(\eta^3) \]
\[ n^m \text{ sat. pushes} \]
\[ n^m \text{ overhead} \]
\[ n^m \text{ related} \]
\[ n^3 \text{ n.s. pushes} \]

\( O(n^3) \) time.

\[ n^2 m \]
**Excess scaling**

\[
e_{\text{max}} = \max \{ e(v) \}
\]

Only push flow from vertices \( v \) if \( e(v) \approx e_{\text{max}} \)

Gradually decrease \( e_{\text{max}} \)

\( \Delta \) an upper bound on \( e_{\text{max}} \)

A node \( v \) with \( e(v) \geq \frac{\Delta}{2} \) has large excess

\( e(v) < \frac{\Delta}{2} \) \( \Rightarrow \) small excess
Invariants to maintain:
1) only push flow from nodes w/ large excess
2) never let $e(v) > \Delta$ for any $v$

\[ e(u) = \frac{3\Delta}{4} \rightarrow \text{push} \frac{3\Delta}{4} - 1 \]

Node selection rule: Choose, among nodes w/ large excess, the one w/ min dist. label

$\Rightarrow$ push from nodes of large excess to small excess
Excess scaling alg
- Preprocess
  - $\Delta = 2^{\lceil \log N \rceil}$
  - while ($\Delta \geq 1$)
    while some node has large excess
      - choose $v$ to be the min dist. labelled node of large excess
        - $w = \text{current-edge}(v)$
          if push($v, w$) is applicable
            - push$_d = \min(e(v), u_f(v, w), \Delta - e(w))$
            - push
          else advance $w$
        else relabel($v$)
    $\Delta = \Delta / 2$
Lemma

1) Each n.s. push sends $\geq \frac{\Delta}{2}$ units
2) No excess ever exceeds $\Delta$.

Pf

2/ Sending min $e(v), \Delta - e(w)$
   but $e(v) \geq \frac{\Delta}{2}$, $e(w) < \frac{\Delta}{2}$
   $\Delta - e(w) \geq \frac{\Delta}{2}$.
Lemma \( O(n^2) \) n.s. pushes per \( \Delta \)-scaling phase.

\[ e_{\text{max}} \leq \alpha \quad \text{many} \quad e_{\text{max}} \leq \alpha/2 \]

\[ \text{Pf} \quad \phi = \sum_v e(v) \cdot d(v) / \Delta \]

At beginning of a scaling phase

\[ \phi \leq \Delta d(v) e(v) \]

\[ \leq 2n \cdot (1) \leq 2n^2. \]

At end of a scaling phase

\[ d \geq 0 \]
\[ \Phi = \sum_{v} \frac{e(v)d(v)}{\Delta} = \sum_{v} d(v) \frac{e(v)}{\Delta} \]

Relabel: \( \Phi \) increases. If \( d(v) \) increases by \( \alpha \)

\( \Phi \) increases by \( \leq \alpha \). From relabelings

\( \Rightarrow \) Total increase in \( \Phi \) \( \leq 2\alpha^2 \)

Push: sat. push decreases \( \Phi \), does not increase \( \Phi \)

(1 unit from \( d(w) = 2 \))

(10^6 \) decreasing \( \Phi \)

\( d(w) = 1 \)

non-sat push: decreases \( \Phi \)

n.s. push sends \( \delta \sum \frac{D}{2} \) units

Decrease in \( \Phi \) = \( \delta (d(v) - d(w)) \geq \frac{(\Delta \alpha)}{\Delta} \)

\( \geq \frac{(\delta \alpha)}{\Delta} \)

\# n.s. pushes \( \leq 2\alpha^2 - 0 + 2\alpha^2 \)

\( \geq 8\alpha^2 \)

\( \geq 1/2 \)
\(n^2\) relabels
\(nm\) time relabeling
\(nm\) sat. pushes

\([\lg \gamma]\) scaling phases
each one does \(O(n^2)\) n.s. pushes
\(O(n^2 \lg \gamma)\) n.s. pushes

\[\text{Time} = O(nm + n^2 \lg \gamma)\]
FIFO
Excess Scaling
Highest label - in practice is best

Data Structures
Dynamic Tree - stores paths

$O(nm[\log (\frac{n^2}{m}) + 1])$
Push/Relabel works well in practice as long as
- reasonable data structures
- gap heuristic

Every 10m steps
- periodically relabel all vertices via BFS.

Increase running time by \(\approx 10^9\), potentially decrease by a lot.