Carpool Fairness

<table>
<thead>
<tr>
<th>Person</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td></td>
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<tr>
<td>3</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

What's a fair division of driving?

<table>
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<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>1/4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1/3</td>
<td>1/3</td>
<td>1/4</td>
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</tr>
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<td>3</td>
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</tr>
</tbody>
</table>

Each person should drive no more than \( \lfloor \frac{1}{5} \rfloor \) times.
A flow "matches" person to a day

A flow of value 5 is a schedule for 0
all 5 days

Does a flow of value 5 exist?

Use a fractional flow (values in table)

A fractional flow of value 5 exists,

\[ \therefore \text{integral flow of value 5 exists.} \]
Given a graph with integral capacities, if an optimal integral flow exists, then there must exist an integral flow of value \[ X \].
Baseball elimination (Sports w/ers end of season problem)

<table>
<thead>
<tr>
<th>Team</th>
<th>Wins</th>
<th>Games Left</th>
<th>Against</th>
<th>NY</th>
<th>BS</th>
<th>TO</th>
<th>BO</th>
</tr>
</thead>
<tbody>
<tr>
<td>NY Yankees</td>
<td>93</td>
<td>8</td>
<td>-</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Bos. Red Sox</td>
<td>89</td>
<td>4</td>
<td>1</td>
<td>-</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Toronto Blue Jays</td>
<td>80</td>
<td>7</td>
<td>6</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>Baltim. Orioles</td>
<td>86</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Which teams are eliminated, have no chance of winning?

Orioles are eliminated

Bostcan win 93, Yankees lose all games

\[
\frac{93 + 88 + 6}{2} = \min(w(ny), w(tor)) \geq 94.
\]
\[ w_i : \text{wins for team } i \]
\[ g_i : \text{games left for } i \]
\[ g_{ij} : \text{games left between } i \text{ and } j \]
\[ \text{For } R \leq T \quad (T = \{ \text{teams} \}) \]
\[ w(R) = \sum_{i \in R} w_i \]
\[ g(R) = \sum_{i < j} g_{ij} \]
\[ a(R) = \frac{w(R) + g(R)}{|R|} \]

If \( i \in T \), \( R \leq T - \{i\} \), and \( a(R) > w_i + g_i \), then \( i \) is eliminated.
k is not eliminated if

\[ \exists x_{ij} \]

\[ x_{ij} + x_{ji} = g_{ij} \quad \forall i, j \in \mathbb{I} \]

\[ w_k + \sum_{j \in \mathbb{I}}^k x_{kj} \geq w_i + \sum_{j \in \mathbb{I}} x_{ij} \quad \forall i \in \mathbb{I} \]

\[ x_{ij} \geq 0 \quad x_{ij} \text{ int} \]

Some way to set the remaining goals so that $k$ comes out in first
k can win $g_k + w_k$ games, can every other team win at most $g_k + w_k$ games
Claim
If a flow of value $\sum_{i,j} g_{ij}$ exists,
then $k$ is not eliminated

$\text{pf}$

$g$ saturates all edges out of $s$, so each game is "set"
$g$ obeys capacities into sink, so each team $i$ wins at most
an additional $g_k + w_k - w_i$ games

$\therefore$ wins $< w_i + (g_k + w_k - w_i)$ = $g_k + w_k$ games.
Claim

If no flow of value \( \sum_{i \in j} g_{ij} \) exists
then \( k \) is eliminated.

Proof

No flow of value \( \sum_{i \in j} g_{ij} \).

\( \sum_{i \in j} g_{ij} = u(s, V - \{s\}) \) is not the min \( \nu \).

Some other min cut

no \( \infty \) cap edges crossing min cut

\[ u(S, \bar{S}) = \sum_{i \in j \in S} g_{ij} + \sum_{i \in R} w_k + g_k - w_i \]

\[ = \sum_{i \in U \in S} g_{ij} + |R| (w_f + g_k) - w(R) \]
\[
\sum_{(i,j) \in S} g_{ij} + |R| (w_k + g_k) - w(R) \\
g(R) \geq \sum_{(i,j) \in S} g_{ij} > |R| (w_k + g_k) - w(R).
\]

If \((i,j) \in S\) then \(i, j \notin R\)

\[
\frac{w(R) + g(R)}{|R|} > w_k + g_k \iff k \text{ is elim.}
\]