distribution network

GM

plant 1

plant 2

min cost flow problem

workhouses

dealers

u = 100

u = 30

u = 10

u = 10

u = 5

u = 5

C = 50

C = 20

C = -10

C = 150

C = 10
Plane holds p people for each i, j (i < j)

\[
\begin{align*}
p &= 5 \\
b_{13} &= 3 \\
b_{24} &= 3
\end{align*}
\]

\[b_{i3} - b_{i2} - b_{13}\]

1 -> 2 -> 3 -> \ldots -> n

- cost
- need somewhere for unstat, demand to go
- max payments

\[b_{2n}\]

\[m_{ij} - \text{payment for flying from } i \text{ to } j\]

\[m_{13} = m_{12} + m_{23}\]
$f_{12}$ people fly $1 \rightarrow 2$
Cost $-m_{12} f_{12}$

Graph with nodes 1, 2, 3, 4, and edges labeled with costs and variables.
Assumptions
- $\sum b(v) = 0$ if $(v,w) \notin G, (w,v) \notin G.$
- Graph is directed
- Costs/cap integral
- Directed path of infinite capacity between each pair of nodes
Residual Graph

\[ G \]

\[ u_f(v, w) = u(v, w) - c(v, w) \]

\[ u_f(v, w) = f(v, w) \]
Optimality of a Flow:

**Lemma**

A feasible flow $f$ is optimal iff $G_f$ has no negative cost cycles.

**Proof**

$r.p.$ if $G_f$ has a non-cost cycle then $f$ is not optimal

- Send 1 unit around cycle
- Flow is still feasible
- $c \cdot f$ decreases by 2
\[ \leq \text{ if a flow has no neg cycles in } G_f \]

No way to push flow along a cycle and decrease the cost

flow decomposition
For max flow:
Given any flows $f, f', F'$ can be obtained from $f$ via $\leq m+n$ augmenting paths or cycles

(PF: homework)

Aside:
For a sequence of $m$ avg. paths that compute a max flow, alg uses $O(nm)$ avg. paths
For min cost flow:

all feasible flows satisfy the (cost) constraints.

\[ \Rightarrow \text{ difference between any two flows is a collection of cycles} \]

\[ f \]

\[ f' \]

\[ f - f' \]