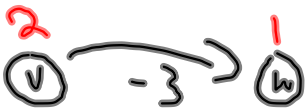
Node potentials $\pi(v)$

reduced cost of edge (v,w) w.r.t π

$$c^{\pi}(v,w) = c(v,w) - \pi(v) + \pi(w)$$



- For any cycle X $\quad \sum_{(v,w)\in X} c^{\pi}(v,w) = \sum_{(v,w)\in X} c(v,w)$

Reduced Cost Opt.

A feasible flow f is optimal iff

$\exists \pi$ s.t. $c^{\pi}(v,w) \geq 0 \quad \forall (v,w) \in G_f$

Pf We need to show that

G_f has no neg. cost cycles $\Leftrightarrow$ $\exists \pi$ s.t. $c^{\pi}(v,w) \geq 0$
$\forall (v,w) \in G_f$

$(\Leftarrow)$ $c^{\pi}(v,w) \geq 0 \Rightarrow$ for a cycle X $\sum_{(v,w) \in X} c^{\pi}(v,w) \geq 0$

$\Rightarrow \sum_{(v,w) \in X} c(v,w) \geq 0$.

$\Rightarrow$ Pick source, compute s.p. dists. d, let $\pi := -d$.

$d(w) \leq d(v) + c(v,w)$

$\Rightarrow c(v,w) + d(v) - d(w) \geq 0$

$\Rightarrow c(v,w) - \pi(v) + \pi(w) \geq 0$

$\Rightarrow c^{\pi}(v,w) \geq 0$.

Complimentory Slackness

feasible f is optimal $\Leftrightarrow \exists\, \pi$ satisfying

1) if $c^\pi(v,w) > 0$ then $f(v,w) = 0$
2) if $0 < f(v,w) < u(v,w)$ then $c^\pi(v,w) = 0$
3) if $c^\pi(v,w) < 0$ then $f(v,w) = u(v,w)$

for all $(v,w) \in G$

Pf

$c^\pi(v,w) \geq 0 \quad \forall (v,w) \in G_f \Rightarrow 1,2,3$

$c^\pi(v,w) = -c^\pi(w,v)$

1) $c^\pi(w,v) < 0 \Rightarrow (w,v) \notin G_f$
$\Rightarrow f(v,w) = 0.$

$v \longrightarrow w$

3) $c^\pi(v,w) < 0 \Rightarrow (v,w) \notin G_f$
$\Rightarrow f(v,w) = u(v,w)$

2) both (v,w) & $(w,v) \in G_f \Rightarrow c^\pi(v,w) \geq 0$ & $c^\pi(w,v) \geq 0$
$\Rightarrow c^\pi(v,w) = 0$

f, π

- Given f, compute π
  s.p. $\pi = -d$.

- Given π, compute f.
  - fix f, on edges w/ $c^{\pi}(v,w) < 0$ & $c^{\pi}(v,w) > 0$
  - solve resulting max flow problem on edges w/ $c^{\pi}(v,w) = 0$.

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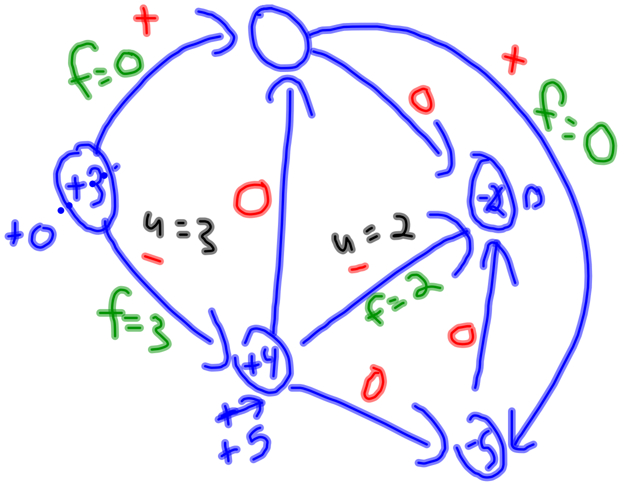

Solve a max flow

Algs

1. Neg Cycle Opt. – Cycle Cancelling algs.
2. Reduced Cost Opt ~ succss. Shortest Path Algs.
3. Comp. slackness ~ Out of Kilter Algs.
4. Network Simplex
5. Push/Relabel Algs.

⋮

Cycle Cancelling

1. Find a feasible flow f (max flow problem)
2. While $\exists$ a neg cost cycle X in G_f
   - let $\delta = \min_{(v,w) \in X} u_f(v,w)$
   - send δ units of flow around X.

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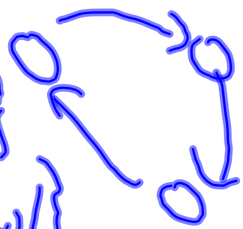
Each iteration decrease obj. by at least 1



$U = \max\{u(v,w)\}$

$C = \max\{|c(v,w)|\}$

cost(cycle) ≤ -1

obj decreases by $\delta \ge 1$

for any feas. flow

$-mCU \le c \cdot f \le mCU$

iterations $\le 2mCU$

$O(nm^2CU)$

Klein's alg.

Improvements

1) Send flow around most negative cycle. NP-hard problem

iterations

Difference between any 2 flows is the union of at most m cycles

f $\quad\quad\quad$ f^*

$f^* - f \leq m$ cycles

most negative cycle must have cost at least

$$\frac{1}{m} \; c\cdot(f^* - f)$$

$$mCU\left(1-\frac{1}{m}\right)^x \approx 2$$

$$\left(1-\frac{1}{m}\right)^x \approx \frac{2}{mCU}$$

iterations $\approx -\lg_{(1-\frac{1}{m})} mCU$

$\frac{1}{m}$ of the way there

How many iterations until $\frac{1}{e}$ way there

$$\left(1-\frac{1}{m}\right)^{x} = \frac{1}{e}$$

m iterations get you $\frac{1}{e}$ of the dist. to opt

$$x \approx$$

$$\left(1-\frac{1}{m}\right)^{m} \approx e^{-1}$$

$O(m \lg(mCU))$ iterations

Send flow around the min. mean cycle.