**Min Mean Cycle Cancelling**

1. Find a feasible flow $f$ (max flow problem)

2. While $\exists$ a negative cycle in $G_f$
   - let $x$ be the min. mean cycle
   - let $\delta = \min_{(v,w)} u_f(v,w)_{\text{net}}$
   - send $\delta$ units of flow around $X$
     (maintain $y_i$'s at nodes)
- $\varepsilon$-optimality
- Flows are always feasible (in this alg.)

A flow is $\varepsilon$-opt. if $\exists \pi$ s.t.

$$C^{\pi}(v,w) \geq -\varepsilon \quad \forall (v,w) \in G_f.$$ 

**Lemma**

1) Any flow is $C$-optimal
2) If $\varepsilon < \frac{1}{n}$, then an $\varepsilon$-optimal flow is optimal.

**Proof**

1) Set $\pi = \emptyset$
2) Consider a cycle $X$ in $G_f$.

$$\sum_{(v,w) \in X} c^{\pi}(v,w) \geq -\varepsilon > -\frac{1}{n}$$

$$|X| \leq n$$

$$\Rightarrow \sum_{(v,w) \in X} c(v,w) \geq 0 \Rightarrow f \text{ is opt.}$$

$$n = \frac{99}{100} - \text{opt.}$$

$$\frac{1}{100} - \text{opt.}$$

$$\frac{1}{100} - \frac{99}{100}$$

$$> \frac{99}{100}$$

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Given \( T \), if let \( \varepsilon^T(f) = \min_{v \in G_f} \left\{ C^T(v,w) \right\} \).

What is the best \( T \)?

Let \( \varepsilon(f) = \min_T \varepsilon^T(f) \).

Let \( \nu(f) = \min_{\text{value}} \text{mean cycle in } G_f \).
Th\textsuperscript{m} \quad \exists(f) = -\mu(f).

Pf \quad 1) \quad \exists(f) \geq -\mu(f)

f \text{ is } \exists(f) \cdot \text{opt.}

\Rightarrow \text{ all cycles } X \subseteq \bigcup_{(u,v) \in X} C(u, v) = \sum_{(u,v) \in X} c(v, w) \geq -\exists(f) \cdot |X|

For min mean cycle \( X' \subseteq \bigcup_{(u,v) \in X'} C(u, v) = \mu(f) \cdot |X'|

\mu(f) \geq -\exists(f) \cdot f)

\mu(f) \leq -\exists(f) \cdot f)
2) \( \varepsilon(f) \leq -\eta(f) \)

Let \( X \) be the minimum mean cycle in \( G_f \).

Let \( C'(v, w) = C(v, w) - \eta(f) \).

With \( C' \), all cycles have non-negative cost.

Pick \( v \), let \( d(v) \) be the shortest path distance from \( v \) to \( v \).

\[ d(w) \leq d(v) + C'(v, w) = d(v) + C(v, w) - \eta(f). \]

Let \( \pi(v) = d(v) \)

\[ C''(v, w) = C(v, w) - \pi(v) + \pi(w) = C(v, w) + d(v) - d(w) = \eta(f). \]

\[ \varepsilon(f) \leq -\eta(f). \]
Lemma. Let $f$ be non-opt. Then $\exists \Pi$ s.t.

$$C^\Pi(v,w) = \mu(f) + -\epsilon(f) A(y,w) X$$

the min mean cycle in $G_f$.

(Previous proof was) $= \text{on min. mean cycle.}$

Diagram: 

- Graph with vertices labeled and directed edges with weights.
Lemma Let $f$ be a non-opt. flow, suppose we push flow around a min. mean cycle $X$ to obtain $f'$. Then $\varepsilon(f') \leq \varepsilon(f)$. 

PF Draw $G_f$ w/ costs $c^\Pi$.

Any new cycles in $G_f$ have at least one edge of $c^\Pi = \varepsilon > 0$, and other edges $c^\Pi \geq -\varepsilon$, and therefore cannot have mean value $\leq -\varepsilon$. 

$\varepsilon \cdot -3$
Lemma: After \( m \) iterations of min. mean cycle
\[
\varepsilon(f') \leq (1 - \frac{1}{n}) \varepsilon(f).
\]
Fix \( \Pi \)

Pf: Case 1: All \( m \) cycles have all edges \( w \)
\[
\Pi(v_{1w}) < 0.
\]
Each iteration removes one edge from \( G_f \)
and replaces it with an edge \( w \)
\[
\Pi(v_{1w}) \geq 0
\]
Total \# of edges \( w \) \( \Pi(v_{1w}) < 0 \) decreased by \( m \)
. \( \varepsilon(f') \) have \( \Pi(v_{1w}) \geq 0 \)
. \( f' \) is optimal.
Case 2: Some minimum cycle has an edge $w$.

$$C^\pi(\{w\}) \geq 0$$

$$n(f') \geq -\frac{3(1X-1)+o(1)}{|1X|}$$

$$\geq -3\left(\frac{n-1}{n}\right)$$

$$\varepsilon(f') \leq \varepsilon(1-\frac{1}{n})$$