Min Mean Cycle Cancelling

1. Find a feasible flow f (max flow problem)

2. While $\exists$ a neg cost cycle in G_f
   - let X be the min. mean cycle
   - let $\delta = \min_{(v,w) \in X} u_f(v,w)$
   - send δ units of flow around X.

   (maintain π's at nodes)

- ε-optimality
- flows are always feasible (in this alg.)

A flow is ε-opt. if $\exists \pi$ s.t.

$$c^{\pi}(v,w) \geq -\varepsilon \quad \forall (v,w) \in G_f.$$

Lemma 1) Any flow is C-optimal

2) If $\varepsilon < \frac{1}{n}$, then an ε-optimal flow is optimal.

Pf 1) Set $\pi = 0$

2) Consider a cycle X in G_f. | $c^{\pi}(v,w) \geq -\varepsilon > -\frac{1}{n}$

$|X| \leq n$

$$\sum_{(v,w)\in X} c^{\pi}(v,w) > \left(-\frac{1}{n}\right)|X|$$

$$> -\frac{1}{n}(n) > -1. \geq 0$$

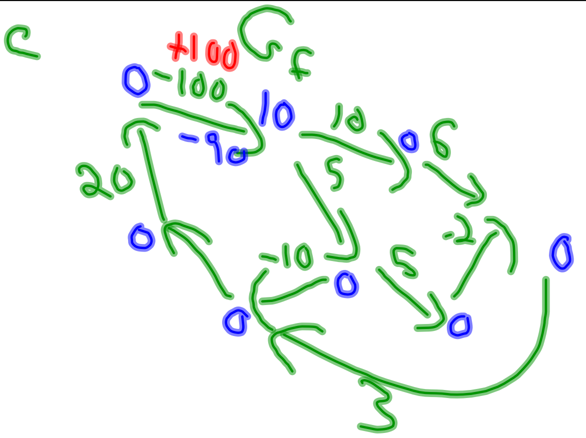
$$\Rightarrow \sum_{(v,w)\in X} c(v,w) \geq 0 \Rightarrow f \text{ is opt.}$$

$n = 99$

$\frac{1}{100}$-opt.

$> -\frac{1}{100}$ $99\left(-\frac{1}{100}\right)$

$> -\frac{99}{100}$



$\varepsilon = 100$

$\varepsilon = 90$

$\varepsilon = 0$

Given π, f let $\varepsilon^{\pi}(f) := \min_{vw \in G_f} \{c^{\pi}(v,w)\}$

· What is the best π?

let $\varepsilon(f) = \min_{\pi} \varepsilon^{\pi}(f)$.

let $\mu(f) =$ min. mean cycle value in G_f

Thm $\varepsilon(f) = -\mu(f)$.

Pf 1) $\varepsilon(f) \geq -\mu(f)$

f is $\varepsilon(f)$-opt.

$\Rightarrow \forall$ cycles X $\sum_{(v,w)\in X} c(v,w) = \sum_{(v,w)\in X} c^{\pi}(v,w) \geq -\varepsilon(f)|X|$

For min mean cycle X' $\sum_{(v,w)\in X'} c(v,w) = \mu(f)|X'|$

$\mu(f) \geq -\varepsilon(f)$

$-\mu(f) \leq \varepsilon(f)$

2) $\varepsilon(f) \le -\mu(f)$

Let X be ~~the~~ a min mean cycle in G_f

let $c'(v,w) = c(v,w) - \mu(f)$

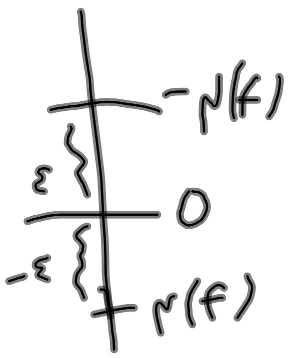
w/r/t c', all cycles have non-neg cost.

Pick v_1 let $d(v)$ be the shortest path distance from v_1 to v w/r/t c'.

$$\Rightarrow \quad d(w) \le d(v) + c'(v,w)$$
$$\le d(v) + c(v,w) - \mu(f).$$

Let $\pi(v) = d(v)$

$$c^{\pi}(v,w) = c(v,w) - \pi(v) + \pi(w)$$
$$= c(v,w) + d(v) - d(w)$$
$$\ge \mu(f)$$



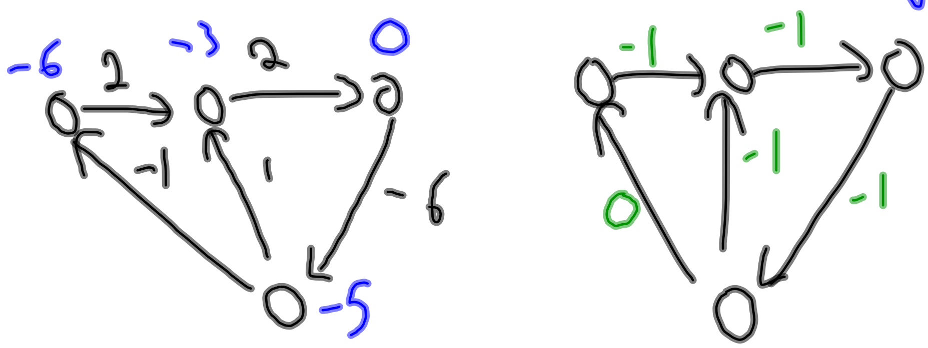
$$\varepsilon(f) \le -\mu(f).$$

Lemma Let f be non-opt. Then $\exists \pi$ s.t.

$$c^{\pi}(v,w) = \mu(f) = -\varepsilon(f) \quad \forall (v,w) \in X$$

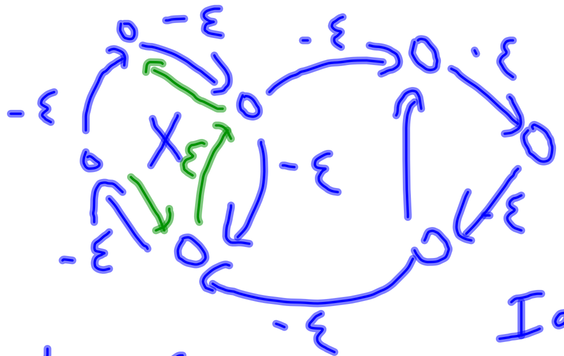
the min mean cycle in G_f.

(Previous proof w/ = on min. mean cycle.



Lemma Let f be a non-opt. flow, suppose we push flow around a min. mean cycle X to obtain f'. Then $\varepsilon(f') \leq \varepsilon(f) =: \varepsilon$

Pf Draw G_f w/ costs c^π



$c^\pi \geq -\varepsilon$

To get f', I delete some edges w/ $c^\pi = -\varepsilon$, I add some edges w/ $c^\pi = \varepsilon$

Any new cycles in $G_{f'}$ have at least one edge of $c^\pi = \varepsilon > 0$, and other edges $c^\pi \geq -\varepsilon$, and therefore cannot have mean value $\leq -\varepsilon$. $-\varepsilon$

$-\varepsilon \cdot \varepsilon \cdot \varepsilon$

Lemma After m iterations of min. mean cycle

$$\varepsilon(f') \le \left(1-\frac{1}{n}\right)\varepsilon(f).$$

f before
f' after

Fix π

Pf Case 1: All m cycles have all edges w/ $c^{\pi}(v,w) < 0$.

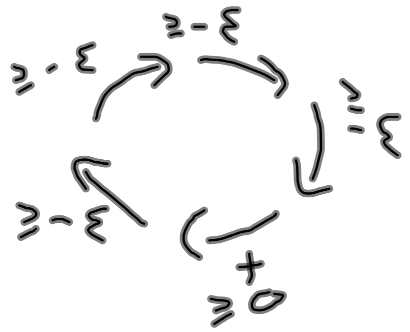
Each iteration removes one edge from G_f & replaces it w/ an edge w/ $c^{\pi}(v,w) \ge 0$

Total # of edges w/ $c^{\pi}(v,w) < 0$ has decreased by m

$\therefore$ all edges in $G_{f'}$ have $c^{\pi}(v,w) \ge 0$,

$\therefore$ f' is optimal.

Case 2: Some min mean cycle has an edge w/ $c^{\pi}(v,w) > 0$



$$\mu(f') \geq \frac{-\varepsilon(|X|-1) + 0(1)}{|X|}$$

$$\geq -\varepsilon\left(\frac{n-1}{n}\right)$$

$$\varepsilon(f') \leq \varepsilon\left(1 - \frac{1}{n}\right)$$