m iterations \in O((1-\frac{1}{n})^m)\approx e^{-1}

\text{initially } \varepsilon \leq \varepsilon

\text{stop when } \varepsilon < \frac{1}{n}

iterate m iterations \ln(nC)\text{ times}

\#iterations = O(nm \log(nC))

running time = O(n^3 m^2 \log(nC)).
Most min-cost flow algs. explicitly scale a parameter (cost or capacity)

Scaling enforces a minimum # of iterations

Strongly polynomial \[ \text{in } n, m \] polynomial “independent” of \[ U, C. \]
MaxFlow $O(n^3)$ alg.
LP, we don't know a S.P. alg.
MinCost has S.P. alg.

Min Mean Cycle is strongly polynomial $O(n^2 m^3 \log n)$ time.
Stronly poly alg

IF $|\mathcal{P}(vw)| \gg \varepsilon(f)$ then $(v,w)$ is fixed

An arc is $\varepsilon$-fixed if the flow is the same for all $\varepsilon'$-opt. flows, $\varepsilon' \leq \varepsilon$.

Once an edge is $\varepsilon$-fixed, we can "freeze" the flow and ignore the edge.
Lemma: If $|C(v,w)| \geq 2n \cdot \varepsilon(f)$, then $(v,w)$ is $\varepsilon(f)$ fixed.

\[\text{Proof:}\]

Suppose $(v,w) \geq 2n \cdot \varepsilon(f)$. Suppose I push flow on $vw$ in cycle $X$.

\[\sum_{v \in X} C(v,w) \geq 2n \varepsilon + (n-1) \cdot (\varepsilon(f))\]

\[\Rightarrow X \text{ is not a min. cost cycle } \Rightarrow \text{new pushed flow.}\]
2) \( c^*(uw) \leq -2 \forall e \)

\[ \exists uvw \in G_f \]

To change flow on uv, I push on

\[ uNv, \text{ but } c^*(uw) = c^*(vw) = 2 \forall e \]

and by case 1, I don't push flow

on uv.
Lemma Every \( nm(\ln n + 1) \) iterations, an edge becomes \( \varepsilon \)-fixed

\[
Pf \quad \begin{align*}
\text{Before } f & \in \Pi \\
\{ \text{nm}(\ln m + 1) \text{ iterations} \} & \\
\text{After } f' & \in \Pi' \\
\varepsilon' & \leq \frac{\varepsilon}{\ln n + 1} = \frac{\varepsilon}{\ln n} \leq \frac{\varepsilon}{2n} \\
\varepsilon' & \leq \frac{\varepsilon}{2n}
\end{align*}
\]

\text{# iterations: } \eta m^2 (\ln n + 1) \\
\eta^2 m^3 (\ln n + 1) \\
\text{running time.}
Look at $X$, a cycle cancelled in the first iteration

\[ \mu(X) = 3 \]

$C^\pi(vw) = 3 \quad \forall (v,w) \in X$

$\sum_{vw \in X} C^\pi(vw) = 1 \times 1$

$\sum_{vw \in X} C^{\pi'}(vw) = 1 \times 1$

$\Rightarrow \nu w$ is $\exists'$ fixed.
New Alg.

 Repeat
 Send flow on a shortest path in $G_f$

 +1

 cost = 0

 not polynomial time
Maintain a pseudoflow $f$

1) $0 \leq f(uv) \leq u(uv)$

$$e(v) = b(v) + \sum_w f(wv) - \sum_{w} f(uw)$$

$e(v)$ is unrestricted

If $e(v) = 0 \ \forall v \in V$ then a pseudoflow is a flow.

r.c. “opt.” of a pseudoflow $f$,

$E \in P$ s.t. $c^E(uv) \geq 0 \ \forall (uv) \in E_f$
Either
1) Assume \( c(vw) \geq 0 \) and \( A(vw) \)
OR
2) Start by setting \( f(vw) = u(vw) \)
for all \( vw \): \( c(vw) < 0 \). Update accordingly.
Lemma: Let $f$ be a pseudoflow sat. $r$,$s$,$o$. with $\Pi$

Let $d(v)$ be s.p. dist. from some node $s$ in $G_f$
$\forall v \in T \ c_{\Pi}(v)$.

Then
1) $f$ satisfies r.c.o. with $\Pi' = \Pi - d$
2) $c_{\Pi'}(vw) = 0$ if $(vw)$ is in a shortest path tree.

Proof:

1) $c_{\Pi}(v,w) = 0 \ \forall (v,w) \in G_f$

2) $d(w) \leq d(v) + c_{\Pi}(v,w) \ \forall (v,w) \in G_f$

$c_{\Pi'}(v,w) = c(v,w) - \Pi'(v) + \Pi'(w)$

$= c(v,w) - \Pi(v) + d(v) + \Pi(w) - d(w)$

$\geq 0$.
Cor. At all points in the successive s.p. alg, \( f \) satisfies f.c.o.

Pf. Only steps to check are

\[ \Pi = \Pi - d \]

Sending flow along \( P \), by (2) of lemma, all edges in \( P \) have \( \Delta \Pi(w,v) = 0 \).

\[ \therefore \] all new residual edges (\( w,v \)) has \( c_{\Pi}(w,v) = 0 \).
\[ G_f(\Delta) = \big\{ (v, w) : G_f \uplus \nu : (v, w) \leq \Delta \big\} \]
Running Time
\[ T_S(\Delta) = \sum_{v \in S(\Delta)} e(v) \]

\( \lg U \) scaling phases

Scaling phase (\( \Delta \) fixed)

Tracking \( \min(S(\Delta), T(\Delta)) \)

Assume the min \( S(\Delta) \)

At the end of 2\( \Delta \) scaling phase

\[ S(2\Delta) = \emptyset \Rightarrow \forall v \in S(\Delta), e(v) \leq 2\Delta \]

\[ |S(\Delta)| = \sum_{v \in S(\Delta)} e(v) \leq \sum_{v \in S(\Delta)} 2\Delta \leq 2\Delta \cdot |S(\Delta)| \]
Next, we saturate edges \( uv \)

\[
C^T(u,v) < 0 \quad \text{or} \quad \eta_f(u,v) \neq 0
\]

This increases \( e(w) \) by \( \eta_f(u,v) \).

\[\text{cost} \quad \Rightarrow \quad \text{increase} \quad e(w)\]

But \( \eta_f(u,v) \leq \Delta \) because \( (u,v) \in \mathcal{G}_f(2\Delta) \)

Each edge saturation can increase \( |S(\Delta)| \) by \( 2\Delta \). Total increase is \( \leq 2m\Delta \)
When we get to the inner while loop
\[ |S(\Delta)| \leq 2^{(n+m)\Delta} .\]

Each augmentation (sending flow along a shortest path) decreases
\[ |S(\Delta)| \] by at least \( \Delta \).

\[
\therefore \text{# iterations} \leq \frac{2^{(n+m)\Delta}}{\Delta} = 2^{(n+m)} \Delta \\
\therefore O(m)
\]
\[ \lg U \]

\[ m \]

\[ m + S.P. \]

\[ (c^T u, v \geq 0) \]

\[ \text{Dijkstra} \]

\[ O(m^2 \lg n \lg U) \quad \text{time} \]

\[ O(m(m+n\lg n)\lg U) \]
Min Cost Flow
Cycle Cancelling
Cap. Scaling (shortest paths)
Cost Scaling (High level view)
iteration
  - establish comp. slackness
  - run a max flow on 0-cost edges.
Cost Scaling

ε-opt. $C^\Pi(v,u) \geq \varepsilon \cdot A^\Pi(v,u)$

Alg

$\Pi = 0$, $\varepsilon = C$

f feasible flow

while ($\varepsilon \geq 1/\nu$)

improve-improve($\varepsilon$, f, $\Pi$)

$\varepsilon = \varepsilon / 2$
Improve-approx
Converts an
3-opt. pseudoflow
to an \( \frac{3}{2} \)-opt. pseudoflow

- Do some push/relabel like, where
  the IT values are your
  "distances"
\[ \Pi(v) \xrightarrow{C(v,w)} \Pi(w) \xrightarrow{0} 0 \]

\[ d(w) = d(v) - 1 \text{ to push in next flow} \]

\[ \Pi(v) - \Pi(w) \geq C(v,w) \text{ to push flow} \]

Some relationship