m iterations $\varepsilon \downarrow$ by $(1-\frac{1}{n})$

$\underline{nm}$ iterations $\varepsilon \downarrow$ by $(1-\frac{1}{n})^n \approx e^{-1}$

initially $\varepsilon \leq C$

stop when $\varepsilon < \frac{1}{n}$

iterate nm iteration $\ln(nC)$ times

$\#$iterations $= O(nm \lg(nC))$

running time $= O(n^2 m^2 \lg(nC))$.

Most min cost flow algs. explicitly scale a parameter
(cost or capacity)

Scaling enforces a minimum # of iterations

Strongly polynomial — polynomial in n, m — polynomial "independent" of u, c.

MaxFlow $O(n^3)$ alg.

LP, we don't know a S.P. alg.

MinCost has S.P. algs.

$\log(nC)$

$m \lg n$

MinMeanCycle is strongly polynomial

$O(n^2 m^3 \log n)$ time.

Strongly poly alg

If $|c^{\pi}(v w)| \gg \varepsilon(f)$ then (v,w) is fixed

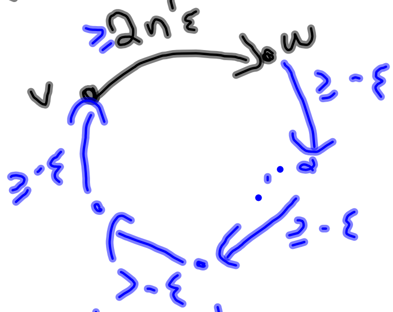
An arc is ε-fixed if the flow is the same for all ε'-opt. flows, $\varepsilon' \le \varepsilon$.

Once an edge is ε-fixed, we can "freeze" the flow and ignore the edge

Lemma If $|c^{\pi}(vw)| \geq 2n\,\varepsilon(f)$, then (v,w) is $\varepsilon(f)$ fixed.

Pf 1) Suppose $c^{\pi}(vw) \geq 2n\varepsilon \quad (\varepsilon = \varepsilon(f))$

Suppose fpoc I push flow on vw in cycle X



$$\sum_{vw \in X} c^{\pi}(vw) \geq 2n\varepsilon + (n-1)(-\varepsilon) = \varepsilon(2n-n+1) = \varepsilon(n+1) > 0$$

$\Rightarrow$ X is not a neg. cost cycle & I never pushed flow.

2) $c^{\pi}(vw) \leq -2n\varepsilon$

$\Rightarrow vw \notin G_f$

To change flow on vw, I push on wv, But $c^{\pi}(wv) = -c^{\pi}(vw) \geq 2n\varepsilon$ and By case 1, I don't push flow on wv.

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Lemma Every $nm(\ln n + 1)$ iterations, an edge becomes ε-fixed

Pf Before f ε π

$\Big\}$ $nm(\ln m + 1)$ iterations

After f' ε' π'

$$\varepsilon' \le \frac{\varepsilon}{e^{\ln n + 1}} = \frac{\varepsilon}{en} \le \frac{\varepsilon}{2n}$$

$$\varepsilon' \le \frac{\varepsilon}{2n}$$

\# iterations
$nm^2(\ln n + 1)$
$n^2m^3(\ln n + 1)$
running time.

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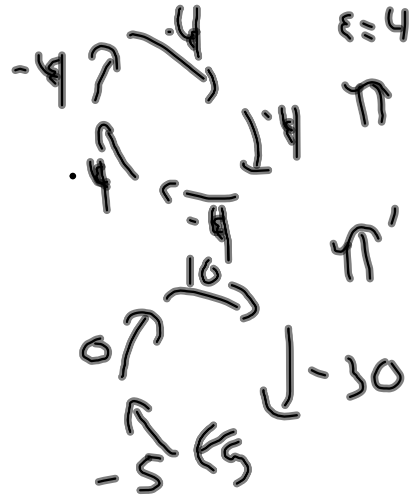
Look at X, a cycle cancelled in the first iteration

$\mu(X) = -\varepsilon$

$c^{\pi}(vw) = -\varepsilon \quad \forall (v,w) \in X$

$\sum_{vw \in X} c^{\pi}(vw) = -\varepsilon |X|$

$\sum_{vw \in X} c^{\pi'}(vw) = -\varepsilon |X|$



$\dfrac{\sum_{vw \in X} c^{\pi'}(vw)}{|X|} = -\varepsilon \Rightarrow$ At least one edge has $c^{\pi'}(vw) \le -\varepsilon \le -2n\varepsilon'$

$\varepsilon' \le \dfrac{\varepsilon}{2n}$

$\varepsilon \ge 2n\varepsilon'$

$-\varepsilon \le -2n\varepsilon'$

$\Rightarrow$ vw is ε' fixed.

lbs. $

Grapes

Cherries

⋮

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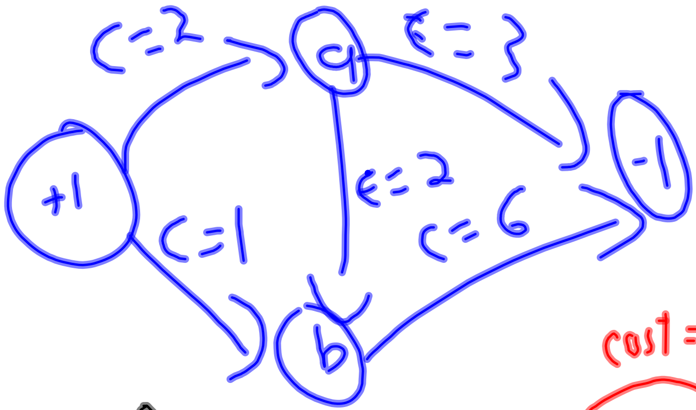
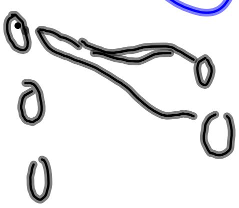
New Alg.





repeat

send flow on a shortest path in G_f

not polynomial time

Maintain a pseudoflow f

1) $0 \leq f(vw) \leq u(vw)$

$$e(v) = b(v) + \sum_w f(wv) - \sum_w f(vw)$$

$e(v)$ is unrestricted

if $e(v) = 0 \ \forall v \in V$ then a pseudoflow is a flow.

r.c. "opt." of a pseudoflow f,

$\exists \pi$ s.t. $c^{\pi}(vw) \geq 0 \ \forall (v,w) \in G_f$

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MMC

f feasible $\rightsquigarrow$ opt.

S.P.

f "optimal" $\rightsquigarrow$ feas.

Either

1) Assume $c(vw) \geq 0 \quad \forall (vw)$

OR

2) Start by setting $f(vw) = u(vw)$ for all vw: $c(vw) < 0$. Update e accordingly.

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Lemma Let f be a pseudoflow sat. r.c.o. w/r/t π

Let $d(v)$ be s.p. dist. from some node s in G_f w/r/t c^{π}.

Then

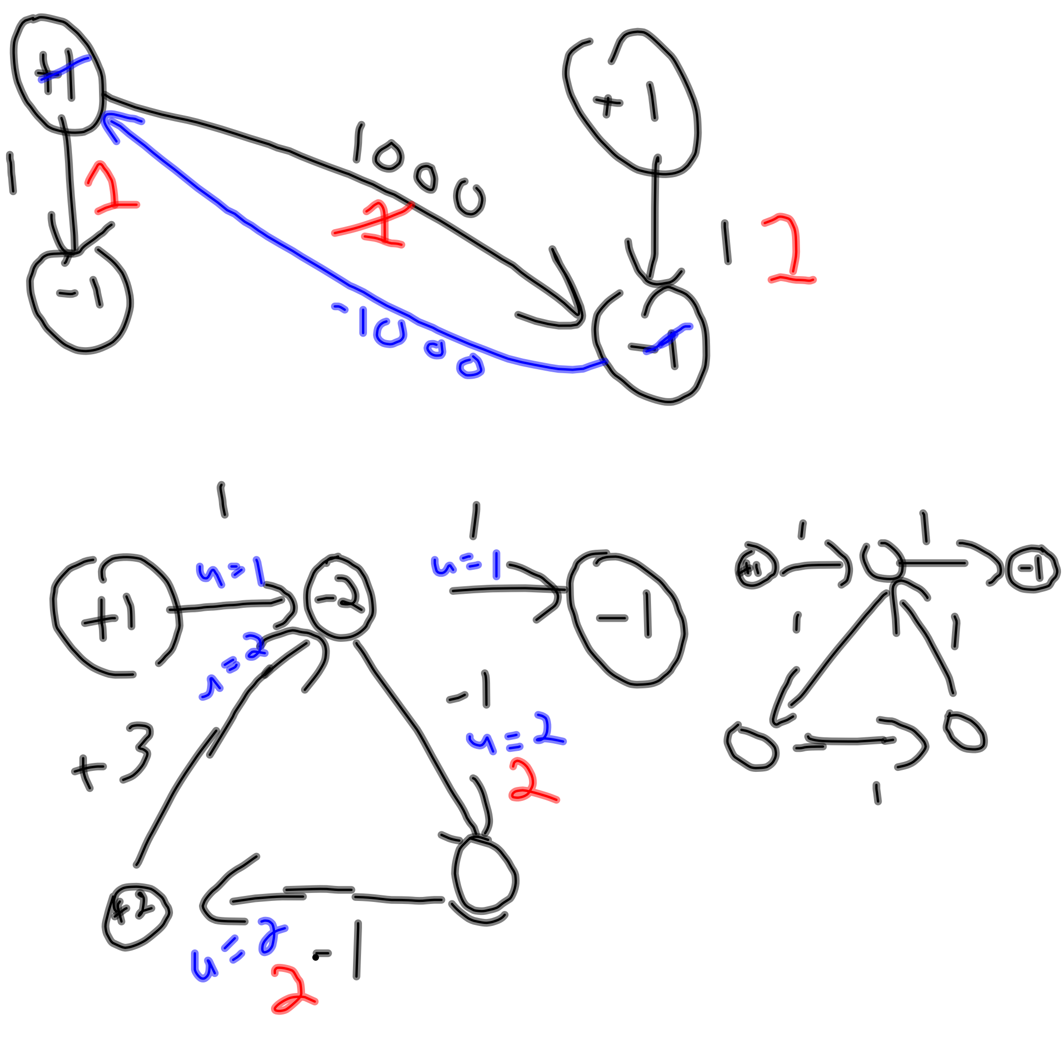
1) f satisfies r.c.o. w/r/t $\pi' = \pi - d$
2) $c^{\pi'}(vw) = 0$ if (vw) is in a shortest path tree.

Pf

$c^{\pi}(v,w) \geq 0 \quad \forall (v,w) \in G_f$

(2) $d(w) \overset{=}{\leq} d(v) + c^{\pi}(vw) \quad \forall (v,w) \in G_f$

$$\begin{aligned} c^{\pi'}(v,w) &= c(v,w) - \pi'(v) + \pi'(w) \\ &= c(v,w) - \pi(v) + d(v) + \pi(w) - d(w) \\ &= c^{\pi}(v,w) + d(v) - d(w) \\ &\overset{=}{\geq} 0. \end{aligned}$$


+1
1
-1
1000
+1
1
-1
-1000
1
+1
u=1
-2
u=2
1
u=1
-1
+3
-1
u=2
+2
u=2
-1
+1
1
1
-1
1
1
1

Cor At all points in the successive s.p. alg, f satisfies F.C.O.

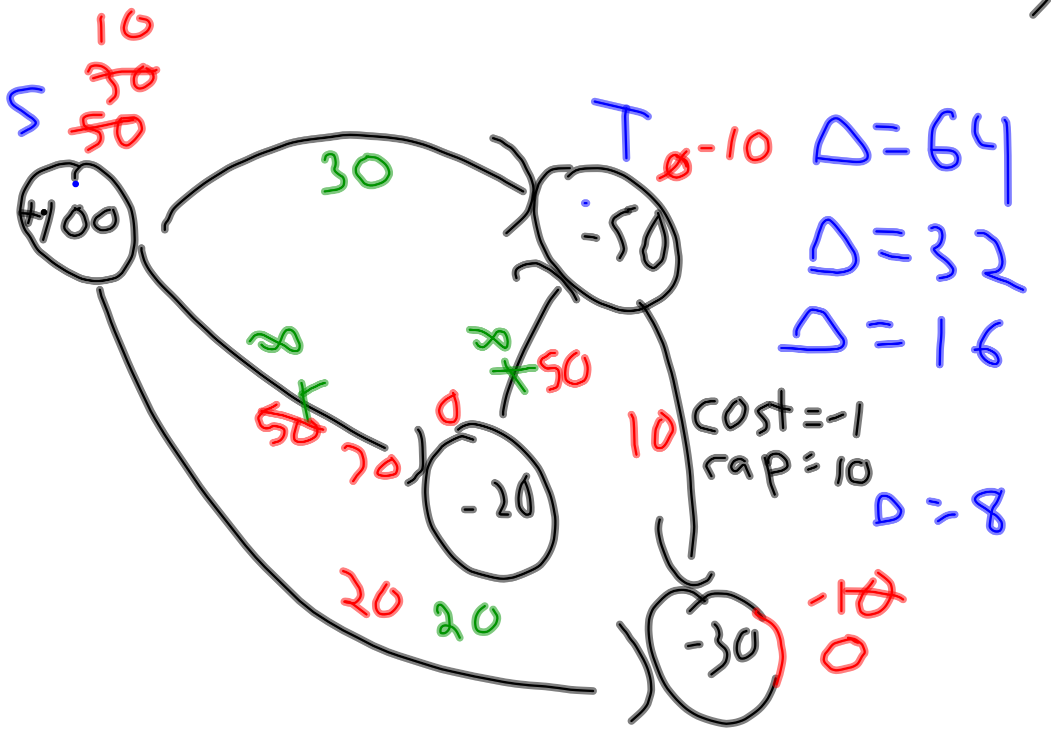
Pf Only steps to check are

$$\pi' = \pi - d \quad \checkmark$$

Sending flow along P,

by (2) of lemma, all edges in P have $c^{\pi'}(v,w) = 0$ $\therefore$ all new residual edges (w,v) has $c^{\pi'}(w,v) = 0$.

$$G_f(\Delta) = \{ (v,w) \in G_f \;\&\; u_f(v,w) \geq \Delta \}$$



Running Time

$|S(\Delta)| = \sum_{v \in S(\Delta)} e(v)$

$\lg U$ scaling phases

Scaling phase (Δ fixed)

Tracking $\min(S(\Delta), T(\Delta))$

assume the min $S(\Delta)$

At the end of 2Δ scaling phase

$S(2\Delta) = \emptyset \Rightarrow \forall v \;\; e(v) \le 2\Delta$

$|S(\Delta)| = \sum_{v \in S(\Delta)} e(v) \le \sum_{v \in S(\Delta)} 2\Delta \le 2n\Delta$

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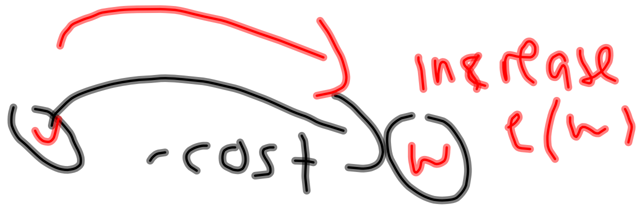
Next, we saturate edges w/

$$c^\pi(v,w) < 0 \quad \& \quad u_f(v,w) \geq \Delta$$

This increases $e(w)$ by $u_f(v,w)$.



But $u_f(v,w) \leq 2\Delta$ because $(v,w) \in G_f(2\Delta)$

Each edge saturation can increase $|S(\Delta)|$ by 2Δ. Total increase is $\leq 2m\Delta$

When we get to the inner while loop

$|S(\Delta)| \le 2(n+m)\Delta$.

Each augmentation (sending flow along a shortest path) decreases $|S(\Delta)|$ by at least Δ.

$\therefore$ # iterations $\le \dfrac{2(n+m)\Delta}{\Delta} = 2(n+m)$

$= O(m)$

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$\lg U$

$\cdot\, m$

$\cdot\, m +$ S.P.

$(c^{\pi}(v,w) \geq 0)$

Dijkstra)

$O(m^2 \lg n \lg U)$ time.

$O(m(m+n\lg n)\lg U)$

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Min Cost Flow

Cycle Cancelling

Cap. Scaling (shortest paths)

Cost Scaling (High level view)

iteration

- establish comp. slackness
- run a max flow on 0-cost edges.

Cost Scaling

ε-opt. $\quad c^{\pi}(v,w) \geq -\varepsilon \quad \forall (v,w) \in G_f$

Alg

$\pi = 0 \quad \varepsilon = C$

f feasible flow

while $(\varepsilon \geq 1/n)$

    implore-implox (ε, f, π)

    $\varepsilon = \varepsilon / 2$

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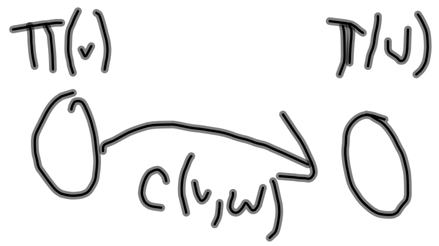
Improve-approx
converts an
ε-opt. pseudoflow
to an $\frac{\varepsilon}{4}$-opt. pseudoflow

- Do some push/relabel like, where
the π values are your
"distances"



$d(w) = d(v) - 1$ to push in max flow

$\pi(v) - \pi(w) \approx c(v,w)$ to push flow

some relationship