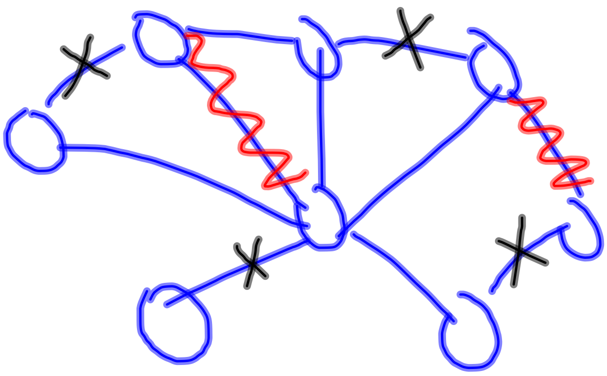
Matchings

Give a graph G, a matching M is a subset of the edges $M \subseteq E$, s.t. each vertex $v \in V$ is incident to at most one edge from M.

is maximal

X is maximum

X is perfect



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Variants

- weighted / unweighted
- bipartite / non-bipartite

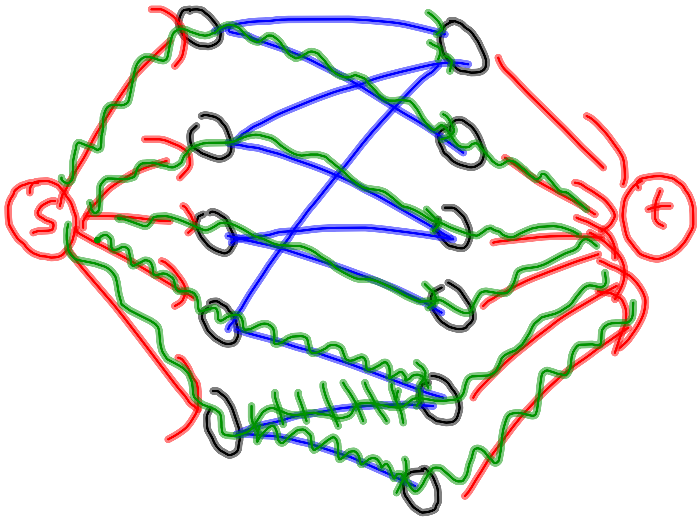
maximum matching $\max |M|$

max wt. matching $\max w(M) = \sum_{uv \in M} w(uv)$

perfect matching $|M| = |V|/2$

maximal matching is a matching M s.t. $\forall e \notin M$, $M \cup \{e\}$ is not a matching.

Bipartite Graphs

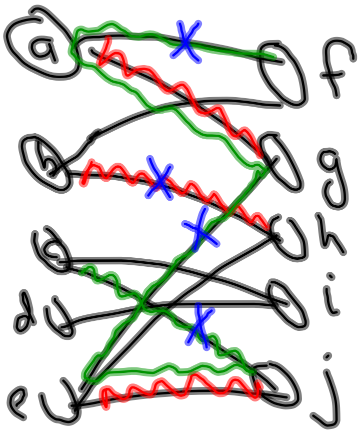


flows ≅ matchings

$O(nm)$ time

a, g, e, j
matched.

b, c, d, f, h, i
free



<u>alternating path</u>

path in which the
edges alternate
between edges in M
& edges not in M.

ag
bh (aug)
fa, ag
cj, je, eg, ga; a f (aug)

matched vertices v (v incident to an edge in M) $v \in M$

free vertices = $\neg$(matched vertices)

augmenting path - alternating path in which the first & last
edges are not in M.
and it starts & ends at a free vertex

shortest aug. path $A \oplus B = (A \cup B) - (A \cap B)$

<u>Fact</u> M matching

P aug. path (relative to M)

Then $M \oplus P$ is a matching

& $|M \oplus P| = |M| + 1$

<u>Fact</u> M matching, $P_1, \ldots, P_k$ are vertex disjoint aug. paths, then

$M \oplus (P_1 \cup \ldots \cup P_k)$ is a matching

of cardinality $|M| + k$

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HK alg

$M = \emptyset$

Repeat

- Let $P = (P_1 \cup \cdots \cup P_k)$ be a maximal set of vertex disjoint shortest aug. paths w.r.t M.
- $M = M \oplus (P_1 \cup \cdots \cup P_k)$

until $P = \emptyset$

$O(nm)$

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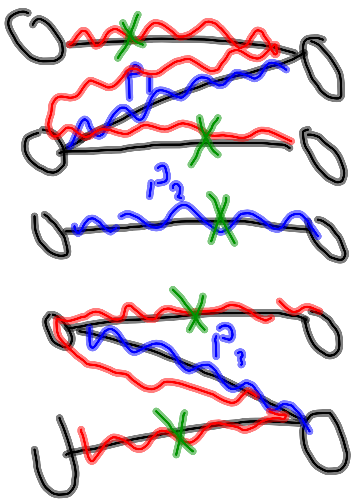

You can find a maximal set of vertex disjoint shortest aug. paths in O(m) time via BFS.

After each iteration, the length of the s.a.p. increases

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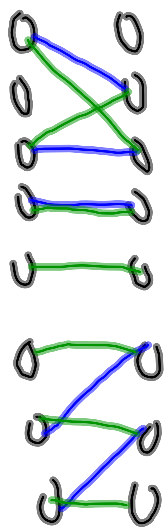
Facts Given two matchings M, M^*

Consider $G' = (V, M \oplus M^*)$

5 7

M M^*



- $\deg(v) \le 2$
- alternating paths & alternating cycles
- If $|M| < |M^*|$, then G' has at least $|M^*| - |M|$ vertex disjoint augmenting paths w.r.t M.

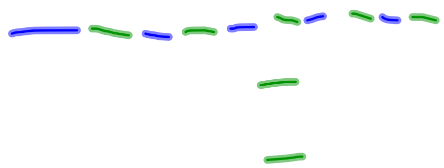
7 green edges

5 blue edges

Draw alt. paths & cycles

minimize the # of paths

starting and ending with green



Lemma Let ℓ = length of s.a.p. w.r.t. M

Let $P_1 \dots P_k$ be a maximal set of vertex-disjoint shortest-augmenting paths

Let $M' = M \oplus (P_1 \cup \dots \cup P_k)$

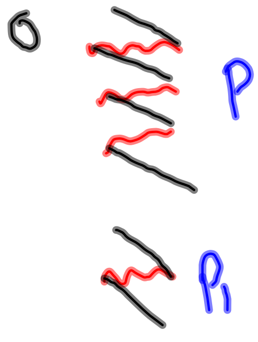
Let P be a s.a.p. w.r.t. M'

Then $|P| > \ell$.

Pf case 1) P is vertex disjoint from $(P_1 \cup \dots P_k) = S$

Then since S is a maximal set, P is not in S,

so $|P| > \ell$.



Case 2) $P \cap (P_1 \cup \dots\dots P_k) \neq \emptyset$

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