Shortest path
Max Flow
Min Cost Flow
We wanted integral optimal solns.

IPs, polynomial time
- Integer programming is NP-hard, but for these problems, we can find an integral opt. soln in poly time.

-T
\[
\max \{ f(s) \} \\
\text{Flow} \leq \text{Flow}_0 \\
f(s) \text{ integral} \\
f(s) \geq 0
\]

Solved the LP
\[ \Downarrow \]
you'd get an integral soln
(basic feasible solns are integral valued)

Unimodular
Def: A matrix \( A \) is \underline{unimodular} if every basis matrix \( B \) of \( A \) has \( \det(B) = \pm 1 \)

(Basis matrix is a \( p \times p \) submatrix w/ linearly ind. columns)

Thm: Let \( A \) be an integer matrix w/ linearly independent rows. Then the following 3 conditions are equivalent:

1) \( A \) is unimodular

2) Every basic feasible solution \( Ax = b \), \( x \geq 0 \) is integral for any integer vector \( b \)

3) Every basis matrix \( B \) has an integer inverse \( B^{-1} \)
\[ \mathbf{A} \mathbf{x} = \mathbf{b} \]

- Graph matrix
- Capacities (supplies, demands)

**Totally unimodular (TU)** A is TU if each square submatrix has determinant  
\[ = -1, 0, 1 \]  

**TU \Rightarrow** unimodular
Thm. The node-arc incidence matrix of a directed network is $TU$. 

$$PF \quad \begin{bmatrix}
\begin{array}{cccc}
-1 & 1 & \cdots & 1 \\
1 & -1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & -1
\end{array}
\end{bmatrix} \quad \begin{bmatrix}
A \cdot f = 0 \\
A \cdot f = ( )
\end{bmatrix}$$

By induction of $k$, the size of submatrix $k=1$.

Inductive step: assume for $k \times k$ submatrices.
\[
\text{node} \{ \ldots \text{AIC} \ldots \} \leq \{ \text{H} \}
\]
Non-bipartite matching

\[ x_{ij} = \begin{cases} 1 & \text{if } (i,j) \in M \\ 0 & \text{o.w.} \end{cases} \]

\[ \max \sum_{(ij) \in E} x_{ij} \]

\[ \text{s.t. } \sum_{(ij) \in E} x_{ij} \leq 1 \quad \forall i \in V \]

\[ x_{ij} \in \{0,1\} \]
Consider $F$ by $(k+1) \times (k+1)$ submatrix

3 cases

1) $F$ has a column of all 0's
2) Each column of $F$ has exactly 2 non-zeroes
3) Some column has exactly one non-zero

1) $\det(F) = 0$

2) Each column has $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

   Sum all the rows, you get $0, 0, \ldots, 0$

   $\Rightarrow$ rows are not linearly independent.

   $\Rightarrow \det(F) = 0$

3) Remove the column with the single $\pm 1$ from $F$ to obtain $F'$, a $k \times k$ submatrix.

   $\det(F') = 0, -1, +1$

   $\det(F) = \pm \det(F') = 0, -1, +1$
Find a graph for which opt. integer soln. ≠ opt. flow integer soln.

\[ \text{max } \sum x_{ij}, \quad \sum x_{ij} \leq 1 \]

not a flow problem

\[ \text{LP} \]

\[ \text{LP} \]

1

1/2

1/2

3/2

1/2
Develop a polynome alg. for matching.
Follow non-bipartite alg. but fix it where it breaks.

**Bipartite case:**

Aug. path

![Diagram of a bipartite graph with labels and paths]

Unique label property: labelling of a node as even/odd is independent of the particular search paths.

An aug. path is an alt path starting at a free vertex, ending at a free vertex, and is labelled odd.

Apr 21-5:02 PM
Lemma: Consider two matchings $M, M^*$ let $A = M\cup M^*$. Connected components of $A$ can be of one of six types.

1) $O \quad M \quad M^* \quad \ldots \quad M^* \quad M$
2) $O \quad M \quad M^* \quad \ldots \quad O \quad M \quad O$
3) $O \quad M^* \quad M \quad \ldots \quad M^* \quad M$
4) $O \quad M^* \quad M \quad \ldots \quad M \quad M^* \quad M$
5) $O \quad M \quad M^* \quad \ldots \quad O \quad M \quad O$
6) $O \quad M \quad M^* \quad \ldots \quad M^* \quad M \quad O$
Lemma (Aug. path) If $p$ is unmatched in a matching $M$, and there is no aug. path starting at $p$, then $p$ is unmatched in some maximum matching.

Proof $M$, consider maximum matching $M^*$. If $p$ is unmatched in $M^*$, we're done. If not, consider $M \oplus M^*$.
\( p \) is unmatched in \( M \), matched in \( M^* \)

- Use Minimal of \( M^* \)
- \( M \) is an aug. path from \( P \), wrt \( M \)
Unique label prop \(\Rightarrow\) we can find an avg. path, if one exists.
stem + blossom = flower

-aug. paths may exist, but may be hard to find

This is the only complication.
Stem: even length alt. path starting at root $p$, ending at $w$  ($p=w$ is possible)

blossom: odd length alt. cycle starting & ending at

Claim: Every node in a blossom is reachable by both an odd and even length alt. path

Idea: label the whole blossom as "even"

Impl. of idea: Contract the blossom
Contract \((v_1, v_2)\) \equiv\ replace \(v_1, v_2\) by a supernode \(v'\) where \(v'\) has an edge to any neighbor of \(v_1\) or \(v_2\).
Aug 12th
1-2-11-9
1-2-3-4-5-9

12-8
1-2-11-8
1-2-3-4-5-6

10-8
1-2-10-8
1-2-3-6-8
Correctness of alg.

\[ G^c \] (contraction graph)
after one blossom

1) Aug path in \( G^c \) \( \Rightarrow \) Aug path in \( G \)
2) Aug path in \( G \) \( \Rightarrow \) Aug path in \( G^c \)

1) To get an aug path in \( G \), take aug. path in \( G^c \) & go around the cycle in appropriate direction
To show
2) If G has an aug path P from p \rightarrow q,
   \text{the } G^c \text{ has an aug path from } p \rightarrow q

Assume p \sim q \text{ are the only unmatched nodes in } G
   \text{let } B \text{ be the blossom}
   \begin{array}{l}
   \text{if } P \cap B = \emptyset \text{ then easy}
   \end{array}

\begin{array}{l}
\text{we consider } P \cap B \neq \emptyset
\end{array}

2 \text{ subcases}
   \begin{array}{l}
   \text{1) node } p \text{ is in } B
   \end{array}
   \begin{array}{l}
   \text{2) node } p \text{ is not in } B
   \end{array}
form a path \( P \) to first int. w\( \beta \)
\( \rightarrow \) last int. w\( \beta \)
\( \rightarrow \) toq.
Polynomial time

$n$ iterations (find. avg. path)

- search for an avg. path
- if find blossom, contract $n$ times

$O(n^2m)$ alg.

$O(n m)$ is possible.