Multicommodity Flow

(set of K commodities

\((s_i, t_i, d_i)\)

some sink demand

capacity constraints are joint cap. constraints.)
variables $f_i(vw)$ flow of commodity $i$ on edge $vw$

\[ \sum_{w} f_i(vw) - \sum_{w} f_i(vw) = \begin{cases} 0 & \text{its origin} \\ \delta_i & \text{its sink} \\ -\delta_i & \text{its tail} \end{cases} \quad \forall(v+w, A) \in K \]

\[ \sum_{k \in K} f_i(vw) \leq u(vw) \quad \forall (vw) \in E \]

\[ f_i(vw) \geq 0 \quad \forall i \in A(vw) \in E \]

Single commodity flow \hspace{1cm} m variables \hspace{1cm} m+n constraints

multi. commodity flow \hspace{1cm} km variables \hspace{1cm} km+n constraints \hspace{1cm} (km non-neg. constr.)

Size of $A$ matrix \hspace{1cm} $m^2$ \hspace{1cm} 1-com. \hspace{1cm} $km (k + m)$ \hspace{1cm} $k^2nm + km^2$

Computationally more \hspace{1cm} 1000 nodes \hspace{1cm} 3000 edges \hspace{1cm} 100 commodities

Challenging problem.
Facts about multicommodity flow

1. LP is big

2. The matrix is not TU.
Opt. solution to a MCF LP may be fractional.

Only feasible sol'n might be fractional

\[ u = 1 \]
\[ d = 1 \]

no integral sol'n

LP infeasible

LP feasible

\[ f_i(u,w) = \frac{1}{2} \]
optimization variants

1. Have $c(uw)$, min cost feasible mcf
   \[ \text{obj } \sum_{i} \leq \sum_{uw} c(uw) \cdot f_i(uw) \]

2. Don't have demands, maximize total flow

3. 1 & 2

4. Send at least 2% of each demand, maximize 2.
Fractional solution is solvable by LP, poly time. Large polynomial ($6^{th}$ degree).

\[ \lambda = \frac{3}{2} \]

2 units from \( s_1 \) to \( t_1 \)
2 units from \( s_2 \) to \( t_2 \)

Maximize \( z \)

\[ z = \frac{3}{2} \]
- no known polytime algos for MCF that do not use LP (comb. algos.)

"easiest" problem w/o a comb. alg.

- comb. algos. that find a \((1+\varepsilon)\)-approx. sol'n in polytime

  \(\text{e.g. } O(3^{-k} \cdot m)\)
Finding a feasible integer soln is NP-complete.

Even disjoint paths problem is NP-complete.

\((u=1, d=1)\)
- Consider the integer problem
  \((u=1, d=1)\)

- Objective \(\max z \text{ s.t. } 2 \cdot \text{fraction of each demand is sent.}\)

  Equivalent problem is send 1 unit of each demand, allow capacity constraints to be violated, but

  \(\min \lambda = \max_{(uv)} \leq f_i(uw)\)
Assume a feasible fractional flow exists. \( (d=1, u=1) \)
- Find integral flow \( \min \lambda \)

**Alg**

1. Find opt fractional flow (via LP)
2. "Round" the fractions

\[ \text{careful about how to round} \]
Rounding

1. Decompose each flow into $s_i t_i$ paths (throw-away cycles) $A$ has paths $p_{ij}$.

2) Interpret the flows $f_{ij}$ as probabilities, choose a path from comm. $i$ according to $p_{ij}$. To get an int flow $g$. 
Analysis

$\lambda = 2$

$g$ satisfies flow-in-flow.

$g$ does not nec. satisfy $\eta$ (minimize $\lambda$).

Show: $\lambda$ is not too large.

Look at one edge. $vw$, think about

$$E\left(\sum g_i(vw)\right)$$

$$\leq \sum f_i(vw) \leq \mu(vw) = 1$$
Use a Chernoff Bound

Let $x_i$ be 0-1 rand. var. $x_i = 1$ w/ prob $p_i$

Let $M = E(\sum x_i) = \sum p_i$. Then for $0 < \beta < 1$

$$\text{Prob}(\sum x_i > (1 + \beta)M) \leq e^{-\beta^2 M/2}.$$

$x_i = g_i$
$p_i = f_i$

$$\beta = \sqrt{\frac{2}{f^*} \ln \left( \frac{n^2}{\epsilon} \right)} \quad 0 < f^* \leq 1$$

$$\lambda \leq f^* + \sqrt{f^* \ln n}.$$