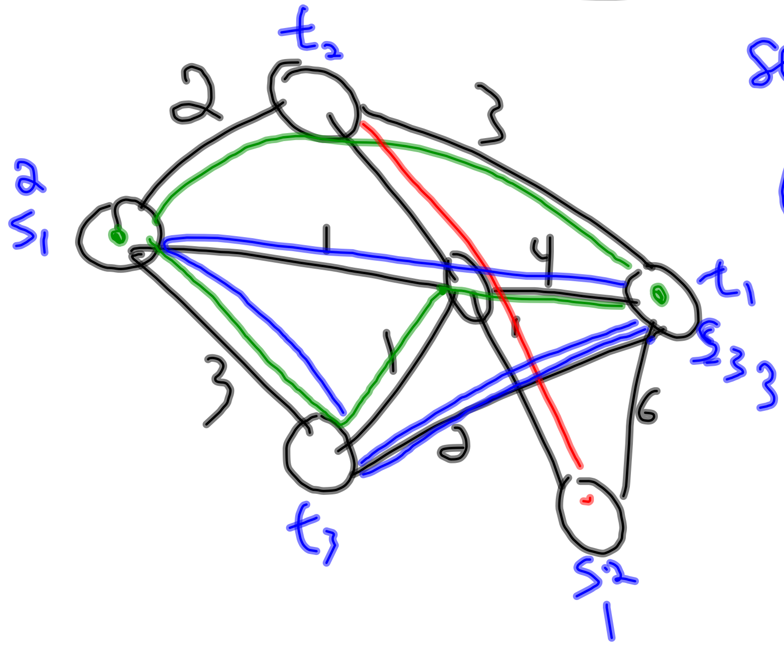
Multicommodity Flow



set of K commodities

(s_i, t_i, d_i)
source, sink, demand

capacity constraints are joint cap. constraints.

variables $f_i(vw)$ flow of comm. i on edge vw
($G=(V,E)$ directed)

$$\sum_w f_i(vw) - \sum_w f_i(wv) = \begin{cases} 0 & i \neq s \neq t \\ d_i & i = s \\ -d_i & i = t \end{cases}$$

$\forall v \in V$ & $\forall i \in K$

$$\sum_{i \in K} f_i(vw) \leq u(vw) \quad \forall (vw) \in E$$

$$f_i(vw) \geq 0 \quad \forall i \ \forall (vw) \in E$$

Single commodity flow — m variables, $m+n$ constraints

multicommodity flow — km variables, $kn+m$ constraints (km non-neg. constr.)

Size of matrix — m^2 1-com.
$km(kn+m)$ $k^2nm + km^2$

computationally more challenging problem.

1000 nodes
3000 edges
100 commodities

Facts about multicommodity flow

1. LP is big
2. The matrix is not TU.

edge × commodity

node × comm.

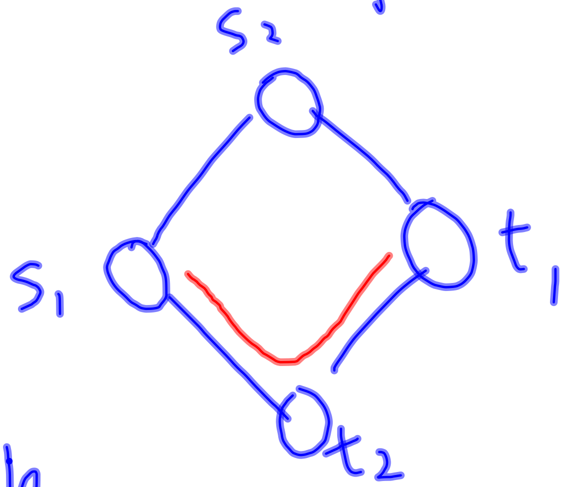
flow in = flow out

cap

not T.U.

Opt. solution to a MCF LP may be fractional.

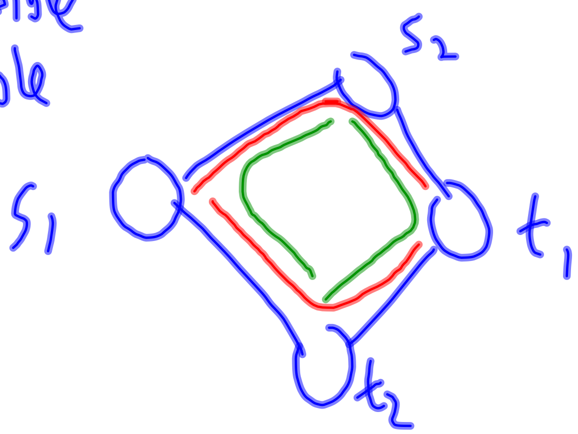
Only feasible sol'n might be fractional



$u=1$
$d=1$

no integral sol'n

IP infeasible
LP feasible



$u=1$
$d=1$
$f_i(vw)=\frac{1}{2}$

optimization variants

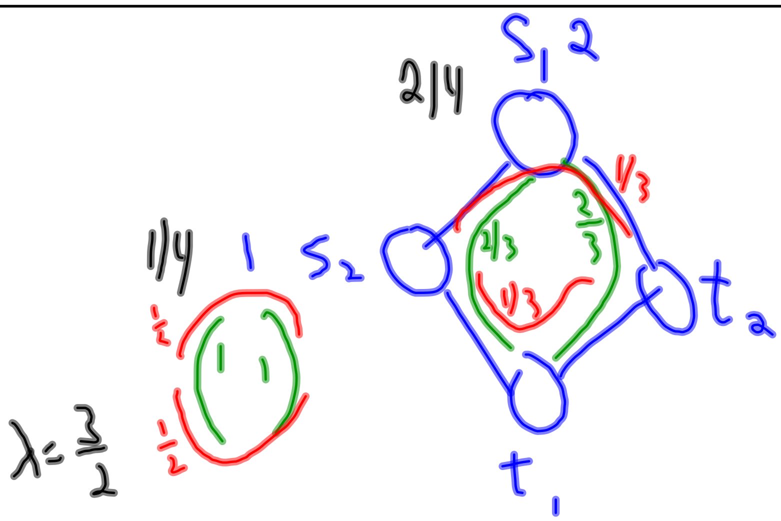
1. Have $c(vw)$, min cost feasible mcf

   obj $\sum_i \sum_{vw} c(vw) \cdot f_i(vw)$.

2. Don't have demands, maximize total flow

3. 1 & 2

4. Send at least z % of each demand, maximize z.



2z units from s_1 to t_1

2 units from s_2 to t_2

maximize z

$z = \frac{2}{3}$

opt. fractional solution is solvable by LP., poly time.

large polynomial (≈6th degree)

$k^2 n^2 m^2$

- no known polytime algs for MCF that do not use LP (comb. algs.)

"easiest" problem w/o a comb. alg.

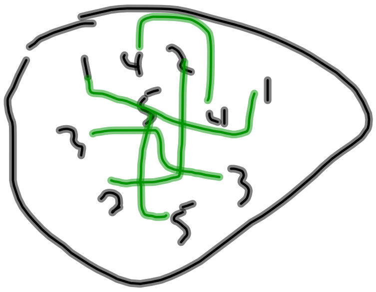
- Comb. algs. that find a $(1+\varepsilon)$-approx. sol'n in poly time
  e.g. $O(\varepsilon^{-2} k n m)$
  ⋮

Apr 28-4:53 PM

Finding a feasible integer soln is NP-complete

even disjoint paths problem is NP-complete

$(u=1, d=1)$



Apr 28-4:57 PM

- Consider the integer problem
  $(u=1, d=1)$
- Objective max z s.t. $z\cdot$ fraction of each demand is sent.

equivalent problem is send 1 unit of each demand, allow capacity constraints to be violate, but

$$\min \lambda = \max_{(vw)} \sum f_i(vw)$$

Apr 28-5:00 PM

Assume a feasible fractional flow exists. $(d=1, u=1)$

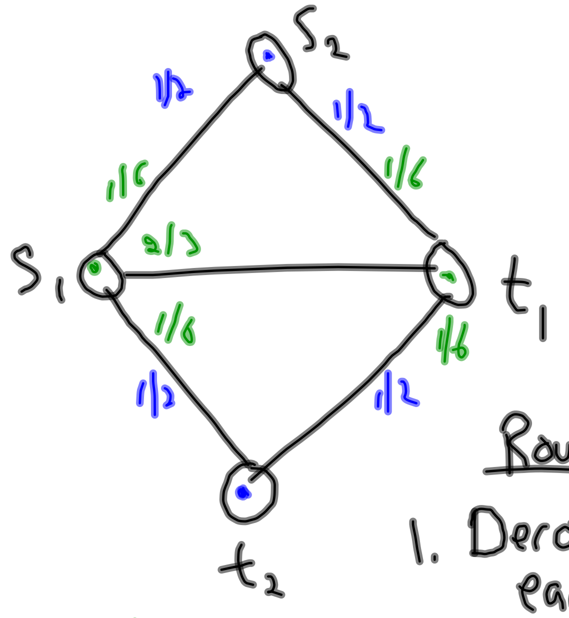
- Find integral flow min λ

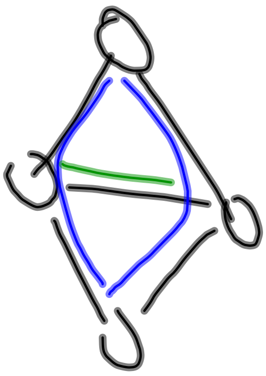
<u>Alg</u>

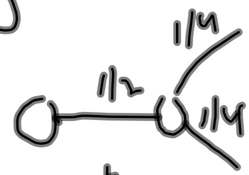
1. Find opt fractional flow (via LP)
2. "Round" the fractions

careful about how to round

Apr 28-5:06 PM

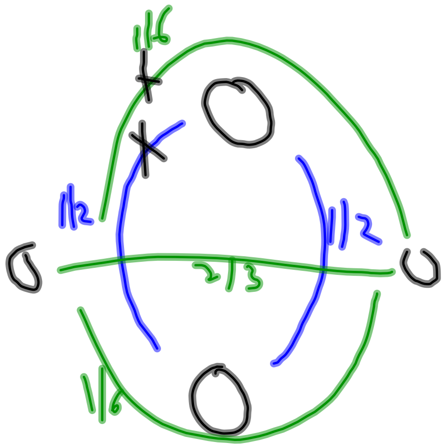






Rounding

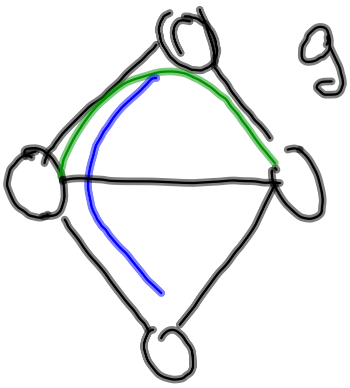
1. Decompose each flow into s_i-t_i paths (throw away cycles).
   comm i has paths P_i^j
   & flows f_i^j



2) Interpret the flows f_i^j as probabilities, choose a path from comm. i according to prob. To get an int flow g

Analysis

$\lambda = 2$



g satisfies flow in = flow out.

g does not necc. satisfies (4) (minimize λ).

Show. λ is not too large

Look at one edge. vw, think about

$$E\left(\sum g_i(vw)\right) = \sum f_i(vw) \le \mu(vw) \ge 1$$

Use a Chernoff Bound

Let x_i be 0-1 rand. var. $x_i = 1$ w/ prob p_i

Let $M = E(\sum x_i) = \sum p_i$. Then for $0 < \beta < 1$

$$\text{Prob}\left(\sum x_i > (1+\beta)M\right) \le e^{-\beta^2 M/2}.$$

$x_i = g_i$

$p_i = f_i$

$$\beta = \sqrt{\frac{2}{f^*} \ln\left(\frac{n^2}{\varepsilon}\right)} \qquad 0 < f^* \le 1$$

$$\lambda \le f^* + \sqrt{f^* \ln n}.$$

Apr 28-5:17 PM