Graph G, capacities $u(v,w)$ of edges (non-neg. ints)
source s, sink t.

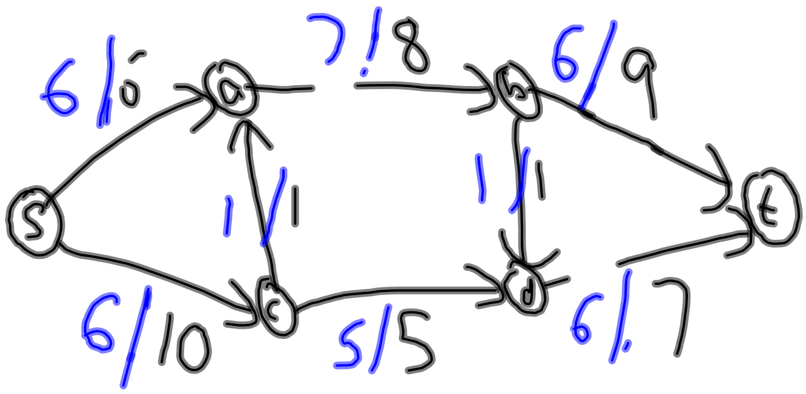
A flow is a real-valued function on edges satisfying

capacity $0 \le f(v,w) \le u(v,w) \quad \forall (v,w) \in E$

flow cons. $\sum_w f(w,v) = \sum_w f(v,w) \quad \forall v \in V - \{s,t\}$

value of a flow $|f| = \sum_v f(v,t) = \sum_v f(s,v)$

If $(v,w) \notin E$, then $f(v,w) = u(v,w) = 0$



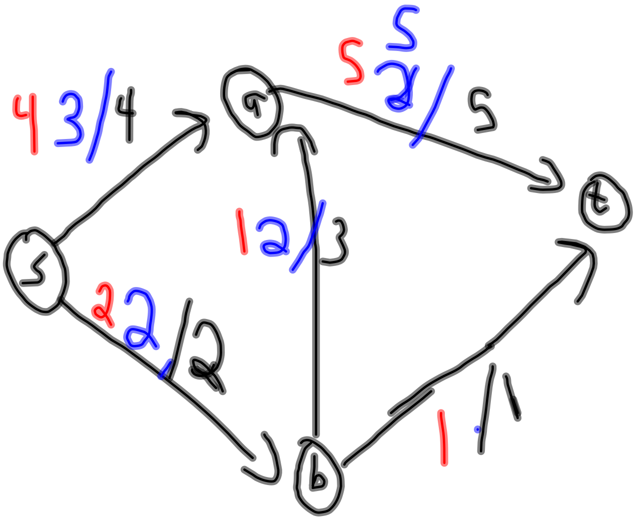
flow
cap.

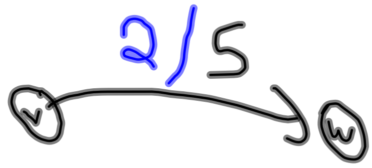
$|f| = \cancel{10}\ \cancel{11}\ 12$

another model : edges have capacity & time

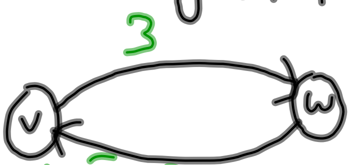
edge cap & a time interval

Residual Graph



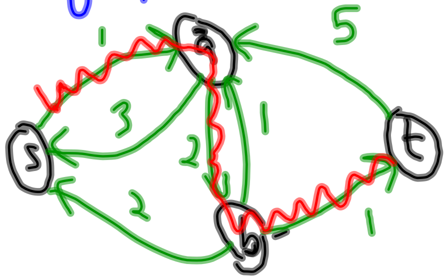


What's the residual capacity of (v,w)

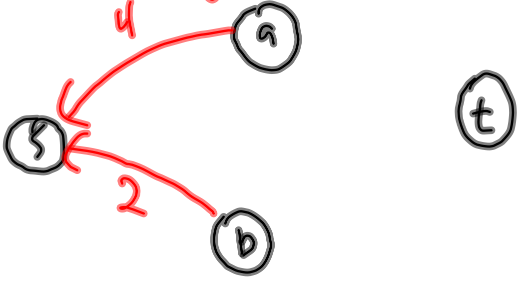


$$u_f(v,w) = \begin{cases} c(v,w) - f(v,w) & \text{if } (v,w) \in E \\ f(w,v) & \text{if } (w,v) \in E \end{cases}$$

Res. graph w/r/t f



Res. graph w/r/t f



Ford Fulkerson "algorithm" $\quad G_f$ = res. graph

$f \equiv 0$

"augmenting path"

While $\exists$ a path p from s to t in G_f

let $\Delta = \min_{e \in p} u_f(e)$

for each edge $(v,w) \in p$

if $(v,w) \in \bar{E}$

$f(v,w) \mathrel{+}= \Delta$

else $f(w,v) \mathrel{-}= \Delta$

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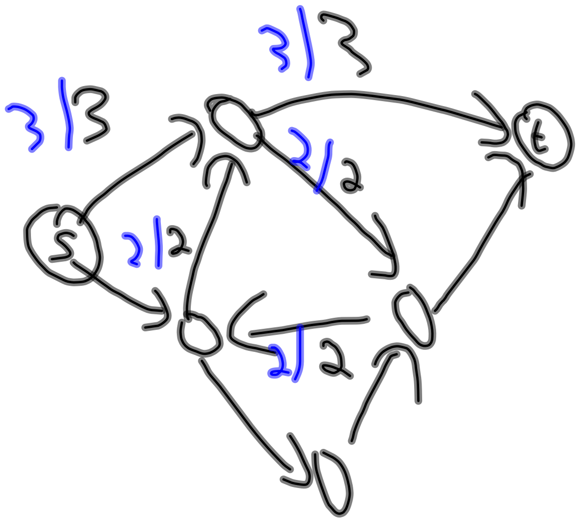
~~s-t cut~~

cycles in flows

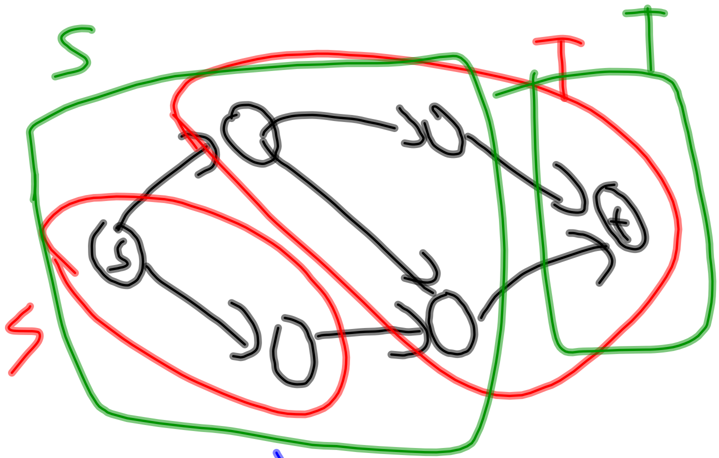


s-t cuts

Partition of vertices into 2 sets S, T

s.t.

$s \in S$
$t \in T$
$S \cap T = \emptyset$
$S \cup T = V$



capacity of a cut

$$u(S,T) = \sum_{\substack{v \in S \\ w \in T}} u(v,w)$$

flow across a cut

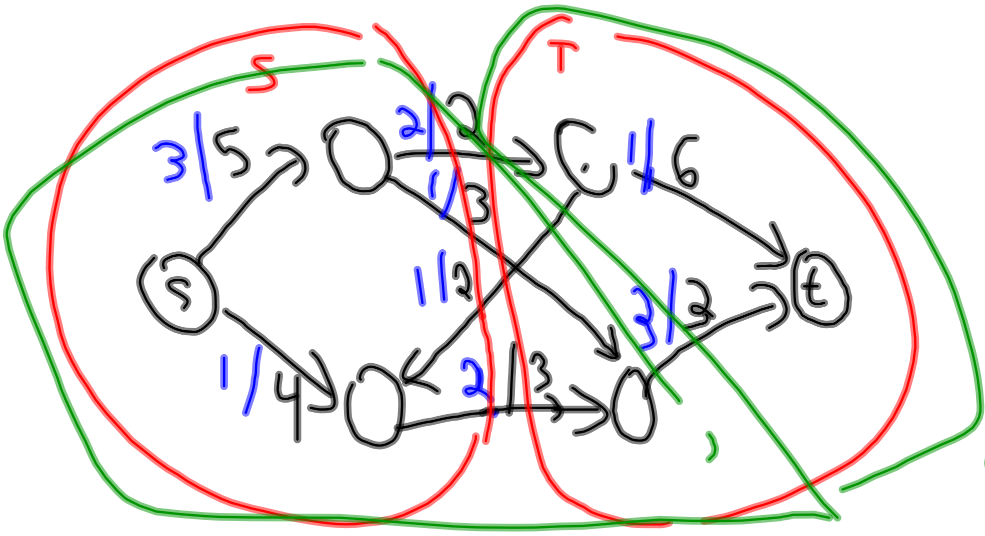
$$f(S,T) = \sum_{\substack{v \in S \\ w \in T}} f(v,w) - \sum_{\substack{w \in T \\ v \in S}} f(w,v)$$

res. cap of a cut

$$u_f(S,T) = u(S,T) - f(S,T)$$

$$\text{mincut} = \min_{\substack{\text{cuts} \\ (S,T)}} u(S,T).$$

(minimum capacity cut)



cuts
5, 8, 9, 9
...

flow
0, 1, 2, 3, 4, 5

$u(S,T) = 2+3+3 = 8$

$f(S,T) = 2+1-1+2 = 4$

cut $s = S$

$u(S,T) = 5+4 = 9$

$f(S,T) = 3+1 = 4$

cut $t = T$

$u(S,T) = 6+3 = 9$

$f(S,T)$ $1+3 = 4$

1) for all cuts (S,T) (S',T'), for any flow

$$|f| = f(S,T) = f(S',T')$$

2) for all cuts (S,T), for all flows f

$$f(S,T) \le u(S,T)$$

3) For all flows, for all cuts

$$|f| \le u(S,T)$$

$\therefore$ $|f| \le$ min cut value

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(FF'56) Max flow = min cut.

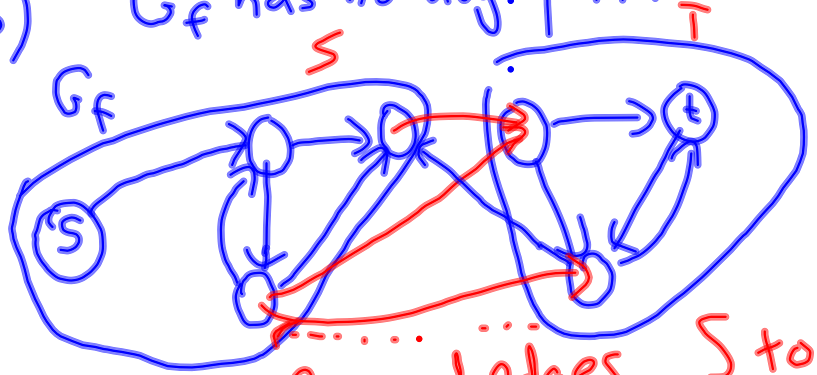
Thm (Given G,) These are equivalent

1) f is a max flow
2) G_f has no augmenting paths
3) $|f| = u(S,T)$ for some cut (S,T)

$(1 \Rightarrow 2)$ $\neg 2 \Rightarrow \neg 1$ G_f has an aug path
$\Rightarrow$ f is not maximum

$(2 \Rightarrow 3)$ G_f has no aug. paths

— res.
— non res. edges



forward edges S to T
$f(v,w) = u(v,w)$
no flow on any edge from T to S in G

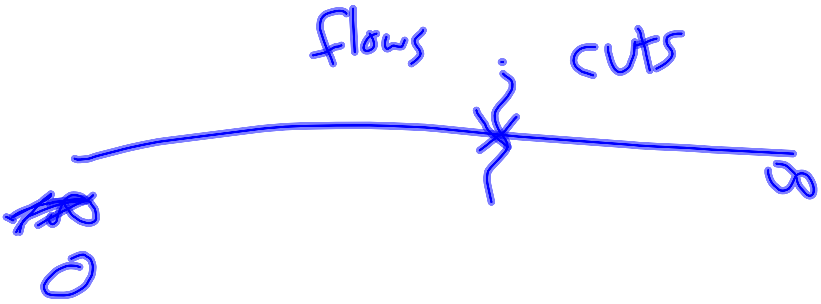
$$|f| = f(S,T) = \sum_{\substack{v \text{ in } S \\ w \text{ in } T}} f(v,w) - \cancel{\sum_{\substack{w \text{ in } T \\ v \text{ in } S}} f(w,v)}$$

$$= \sum_{\substack{v \text{ in } S \\ w \text{ in } T}} u(v,w) = u(S,T)$$

$3 \Rightarrow 1$ $\quad |f| = u(S,T)$ for some (S,T)

we know that $|f| \le u(S',T')$ for any S',T'

$\therefore f$ is maximum



Analysis of F-F.

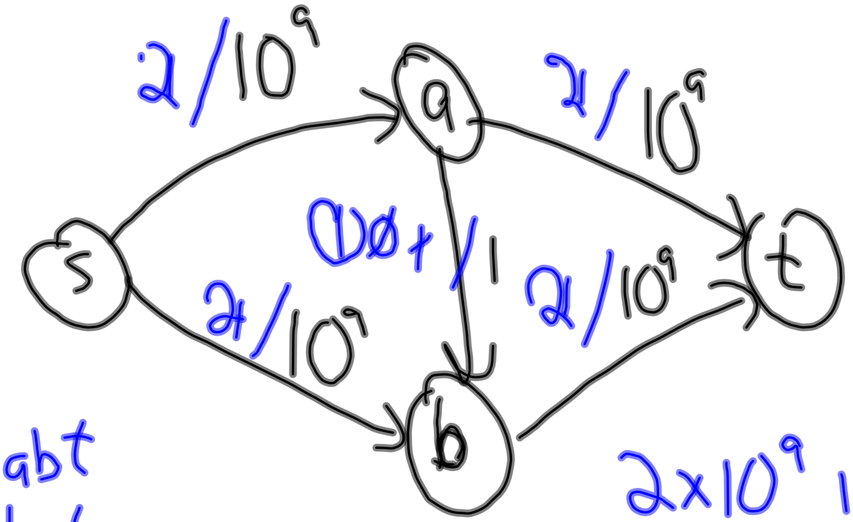
Each iteration sends at least 1 unit of flow from source to sink

$|f^*|$ = val of max flow

$O(|f^*| m)$ alg.

$|f^*| \le nU$

$O(nmU)$ pseudopolynomial



sabt
sbat

2×10^9 iterations
$\sim nU$