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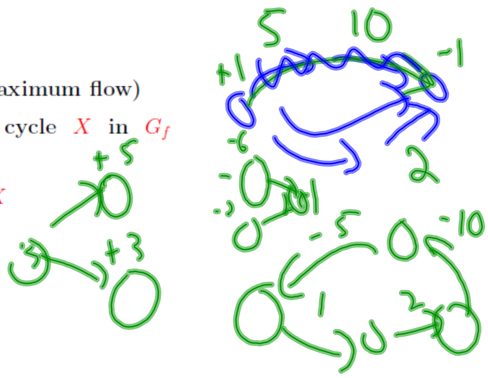
## Cycle Cancelling Algorithm

**Basic Algorithm (Klein's Algorithm)**

- Find a feasible flow $f$ (solve a maximum flow)
- While there exists a negative cost cycle $X$ in $G_f$
  - Let $\delta = \min_{(v,w)\in X} u_f(v,w)$
  - Send $\delta$ units of flow around $X$



**Analysis:**

- Let $U = \max_{(v,w)\in E} u(v,w)$
- Let $C = \max_{(v,w)\in E} c(v,w)$
- For any feasible flow $-mCU \le c(f) \le mCU$
- Each iteration of the Basic Cycle Cancelling Algorithm decreases objective by at least 1.
- Conclusion: At most $2mCU$ iterations.
- Running time = $O(nm^2CU)$ . Not polynomial.

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## Cycle Cancelling Algorithm

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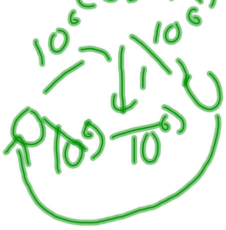
$\delta$ ~~cap~~ $\ge 1$

$\text{cost}(X) \le -1$



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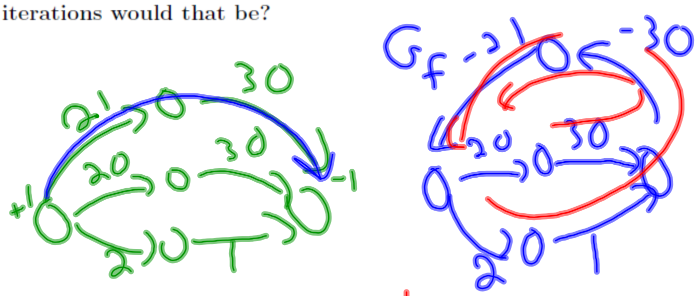
## Ideas for Improvement

– Send flow around most negative cycle. (NP-hard to find)
– How many iterations would that be?




-1
-48

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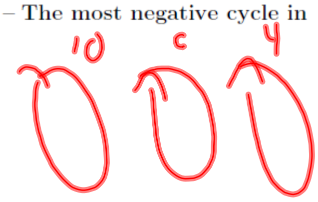
## Ideas for Improvement

– Send flow around most negative cycle. (NP-hard to find)
– How many iterations would that be?

Analysis:

– The difference between any two feasible flows is the union of at most $m$ cycles.
– Let $f$ be the current flow, $f^*$ be the optimal flow.
– Consider $f - f^*$. It is the union of at most $m$ cycles.
20 – The most negative cycle in $f - f^*$ must have cost at least




$\frac{1}{m}c(f^* - f)$

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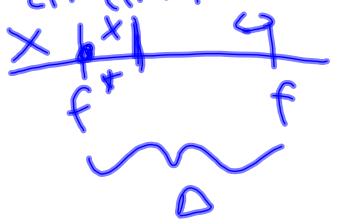
## Analysis continued

- Each iteration gets $\frac{1}{m}$ of the way to the optimal flow.
- Equivalently, each iteration decreases the distance to the optimal flow by a $1 - \frac{1}{m}$ factor.
- Initial distance is at most $2mCU$ .
- Once we get within one of the optimal flow, we are done, since flows, and costs of flows are integers.

$c(f^*)$ $c(f^*)+1$ $\frac{1}{m}\Delta$




**Conclusion:** The number of iterations is

$$\lg_{1-1/m}(mCU)$$

.

**Analysis:**

$$\begin{aligned}\lg_{1-1/m}(mCU) &= \frac{\lg(mCU)}{\lg(1-\frac{1}{m})}\\ &\approx \frac{\lg(mCU)}{\frac{1}{m}}\\ &= m\lg(mCU)\end{aligned}$$

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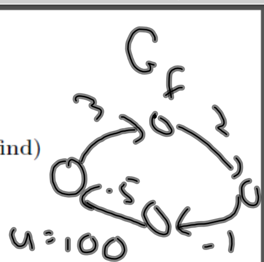
## Ideas for Improvement




- Send flow around most negative cycle. (NP-hard to find)
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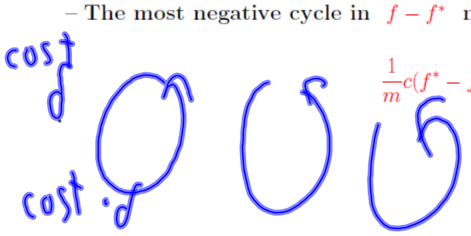
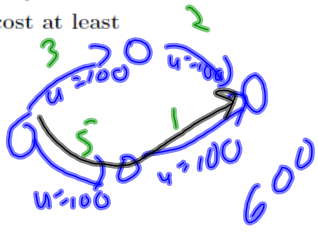
- The most negative cycle in $f - f^*$ must have cost at least

500

$$\frac{1}{m}c(f^* - f)$$

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## Cycle Cancelling

- If we could find most negative cycle, there would be a polynomial number of iterations.
- Finding the most negative cycle is NP-hard.
- Solution: Find minimum mean cycle and cancel it.
- We will show that the minimum mean cycle "aproximates" the most negative cycle well.


if neg cycle ⇒ min mean cycle is negative

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## $\epsilon$-optimality

Lemma:

- Any feasible flow is $C$-optimal.
- If $\epsilon < 1/n$, then an $\epsilon$-optimal flow is optimal.

C-optimal $\exists \pi$ s.t.

$c^{\pi}(v,w) \geq -C \quad \forall (v,w) \in G_f$

(choose $\pi = 0$, this is true)

all edges sat. $-C \leq c(v,w) \leq C$

any edge is $G_f$ sat. $-C \leq c(v,w) \leq C$

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## $\epsilon$-optimality

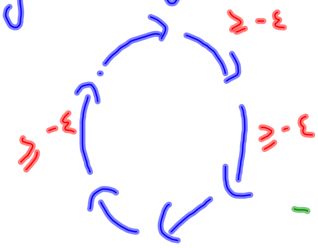
**Lemma:**

- Any feasible flow is $C$-optimal.
- 2) If $\epsilon < 1/n$, then an $\epsilon$-optimal flow is optimal.

2) $f$ is $\varepsilon$-optimal $\Rightarrow c^{\pi}(v,w) \geq -\varepsilon \;\; \forall (v,w) \in G_f$

Assume $f$ is not opt.

Look at a neg. cost cycle in $G_f$




What is the smallest (most neg) value the cycle can have?

$-n\varepsilon > -n\left(\frac{1}{n}\right)$

$> -1 \geq 0.$

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## Main Theorem

$c^{\pi}(v,w) \geq -\varepsilon$

**Defining $\epsilon$ given $f$ and $\pi$:** Given $\pi$ and $f$, let $\epsilon^{\pi}(f) = -\min_{(v,w)\in G_f}\{c^{\pi}(v,w)\}$. This value is the smallest $\epsilon$ for which the flow $f$ is $\epsilon$-optimal.

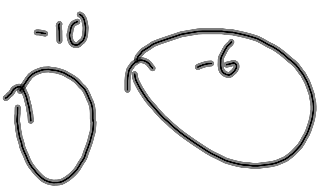
$\varepsilon^{\pi}(f) = 10$

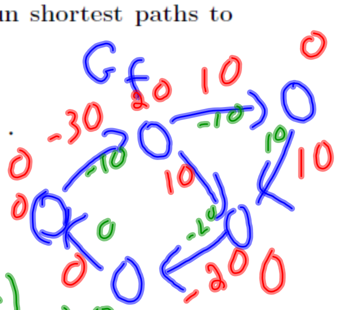
**Choosing $\pi$, given $f$**

- Note that $f$ is not optimal, so we cannot just run shortest paths to find an optimal $\pi$
- Let $\epsilon(f) = \min_{\pi} \epsilon^{\pi}(f)$.
- Let $\mu(f)$ be the minimum mean cycle value in $G_f$.

**Theorem** Given any feasible flow $f$

$$\epsilon(f) = -\mu(f)$$




$\pi = 0$

$\varepsilon^{\pi}(f) = 30$

$\varepsilon^{\pi}(f) = 20$

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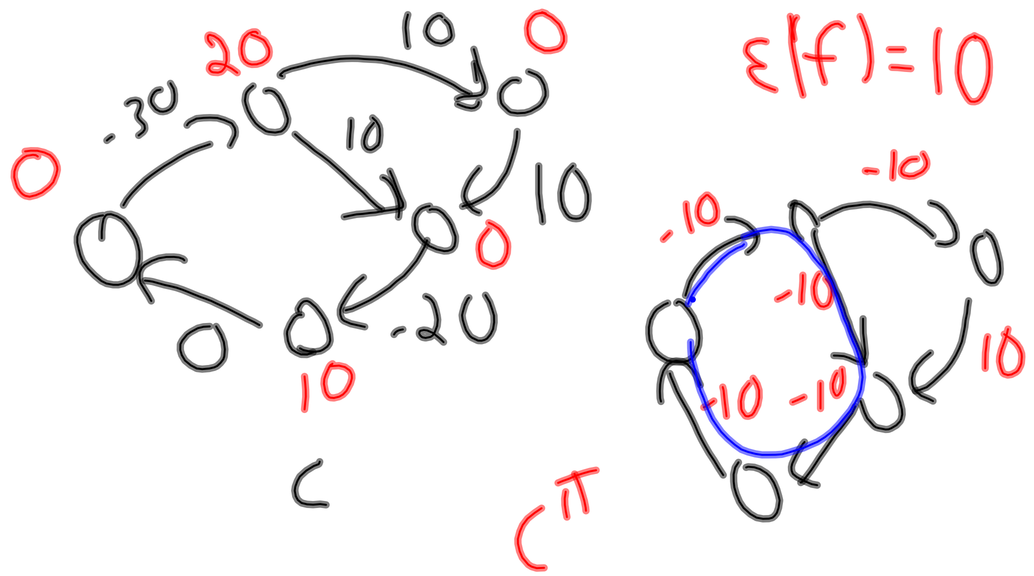


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$$\underline{\varepsilon(f) = -\mu(f)}$$

$|X| = \#$ edges in $X$

1) $\varepsilon(f) \geq -\mu(f)$

Pf $f$ is $\varepsilon(f)$-opt

$\Rightarrow \forall$ cycles $X \in G_f$

$$\sum_{(v,w)\in X} c(v,w) = \sum_{(v,w)\in X} c^{\pi}(v,w) \geq -\varepsilon(f)|X|$$

Min mean cycle $X'$,

$$\sum_{(v,w)\in X'} c(v,w) = \mu(f)\,|X'|$$

Combining $\mu(f)|X'| \geq -\varepsilon(f)\,|X'|$

$\Rightarrow \mu(f) \geq -\varepsilon(f)$

$\Rightarrow -\mu(f) \leq \varepsilon(f)$.

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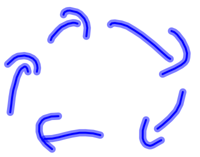
2) $\varepsilon(f) \le -\mu(f)$.

Let $X$ be a min. mean cycle in $G_f$

let $c'(v,w) = c(v,w) - \mu(f)$

w.r.t. $c'$, no neg. cost cycles



if min mean value is $-10$

this cycle has cost $\geq -50$

subtract $-50$ from this cycle $\Rightarrow$ non-negative.

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